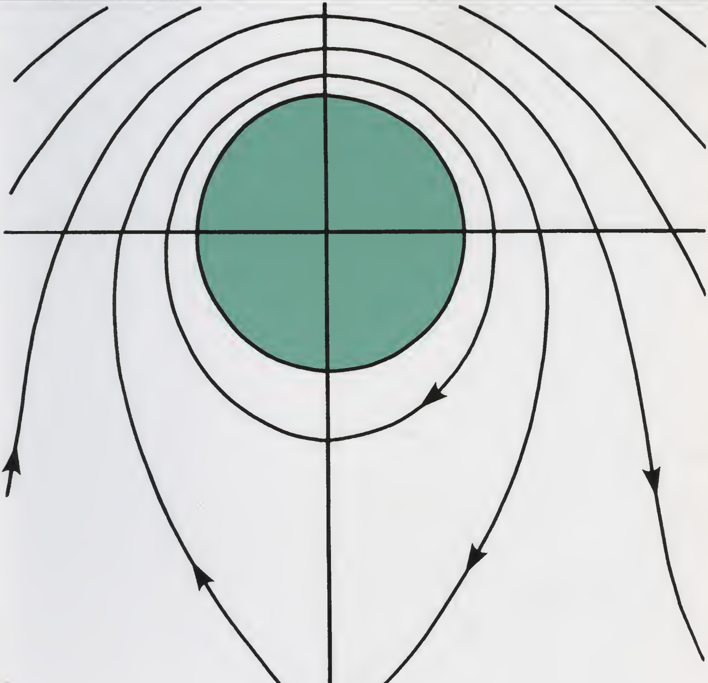
M337 Unit D3
Mathematics. A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT D3 THE MANDELBROT SET



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UNIT D3 THE MANDELBROT SET

Prepared by the Course Team

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Before working through this text, make sure that you have read the *Course Guide* for M337 Complex Analysis.

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1.4

CONTENTS

INTRODUCTION

Many techniques for solving equations involve *iteration*, otherwise known as the method of 'refining guesses'. For example, one way to solve a real polynomial equation $p(x) = 0$ is to use the Newton–Raphson method, based on the recurrence relation:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}, \qquad n = 0, 1, 2, \ldots .$$

The justification of this recurrence relation is shown in Figure 0.1, which indicates that if x_n is close to a zero a of p, then x_{n+1} is (usually) even closer. In particular, if $p(x) = x^2 - 2$, then the Newton–Raphson recurrence relation is

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \tfrac{1}{2}\left(x_n + \frac{2}{x_n}\right), \qquad n = 0, 1, 2, \ldots .$$

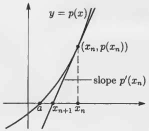

Figure 0.1
$p'(x_n) = \dfrac{p(x_n)}{x_n - x_{n+1}}$

An initial guess of $x_0 = 2$ yields the sequence of refined guesses $2, 1.5, 1.417, \ldots$, which converges rapidly to the known solution $\sqrt{2}$.

In this unit we study the behaviour of *complex* iterative processes of the form

$$z_{n+1} = f(z_n), \qquad n = 0, 1, 2, \ldots,$$

where f is an analytic function. Usually f will be a polynomial function, although we spend a short time in Section 1 discussing the Newton–Raphson method for complex polynomials, which involves the iteration of a rational function. Most of Section 1 is devoted to the basic notions associated with iteration: *nth iterates*, *fixed points* and *conjugate iteration sequences.*

Section 2 takes up the iteration of complex quadratic functions, which is the main topic of the unit. We begin by showing that it is sufficient to consider quadratic functions belonging to the basic family $\{P_c(z) = z^2 + c : c \in \mathbb{C}\}$, because each quadratic function is equivalent, in a certain sense, to one of this form. We then consider, for any given $c \in \mathbb{C}$, the set of points E_c which 'escape to ∞' under iteration of P_c, and establish some basic properties of E_c and its complement K_c. To determine some of the points of K_c, we introduce the idea of a *periodic cycle* of points, and we show that certain periodic cycles must lie in the interior of K_c, whereas others lie on the boundary of K_c. This boundary is called the *Julia set* J_c of P_c (see Figure 0.2).

Figure 0.2 A Julia set

In Section 3, we use a technique called *graphical iteration*, which applies only to real functions, to obtain various properties of the set K_c when c is real. For example, we determine the nature of the set $K_c \cap \mathbb{R}$ for all real numbers c.

In Section 4, we define the *Mandelbrot set*, which is the set M of numbers c such that the set K_c is connected, and we obtain a criterion for c to belong to M. This criterion makes it possible to plot pictures of M (see Figure 0.3) and thus reveal its immensely complicated structure. We investigate this structure by using a result which states that if P_c has a so-called *attracting cycle*, then $c \in M$.

Figure 0.3
The Mandelbrot set

You may be tempted to think that the Mandelbrot set is in some sense an oddity, arising from some special property of quadratic functions. However, in Section 5, we describe briefly some different iteration sequences, and we find that the Mandelbrot set is a truly universal object.

Study guide

You may find some of the ideas in this unit quite challenging. However, your labours will be rewarded by gaining some insight into a fascinating subject, which is the object of current research. Some of this research is mentioned in the discussion on the audio tape, which has no associated frames and which may be listened to at any time.

Section 5 and Subsection 2.4 are intended for reading only.

Associated with this unit is a segment of the Video Tape for the course. Although this unit text is self-contained, access to the video tape will enhance your understanding. Suitable points at which to view the video tape are indicated by a symbol placed in the margin.

1 ITERATION OF ANALYTIC FUNCTIONS

After working through this section, you should be able to:

(a) calculate several terms of a given *iteration sequence*;

(b) determine the *nth iterate* of a given analytic function, for small values of n;

(c) determine the *fixed points* of certain analytic functions and find their nature;

(d) calculate *conjugate iteration sequences*;

(e) determine the Newton–Raphson function N for a given polynomial function p, and describe the iterative behaviour of N when p is a quadratic function.

1.1 What is an iteration sequence?

Any sequence $\{z_n\}$ defined by a recurrence relation of the form

$$z_{n+1} = f(z_n), \qquad n = 0, 1, 2, \ldots,$$

where f is a function, is called an **iteration sequence** with **initial term** z_0.

Some texts call z_0 the *seed* and $\{z_n\}$ the *orbit* of z_0. Also note that the study of iteration sequences is often called *dynamical systems*.

For example, if $f(z) = 2z$, then the recurrence relation is

$$z_{n+1} = 2z_n, \qquad n = 0, 1, 2, \ldots.$$

If $z_0 = i$, then the first few terms of the corresponding iteration sequence are

$$z_0 = i, \quad z_1 = 2i, \quad z_2 = 4i, \quad z_3 = 8i, \ldots.$$

In this case the sequence $\{z_n\}$ tends to infinity, but if we had chosen $z_0 = 0$, then the corresponding iteration sequence would be constant:

$$z_0 = 0, \quad z_1 = 0, \quad z_2 = 0, \quad z_3 = 0, \ldots.$$

Thus the behaviour of a given iteration sequence depends not only on the function f, but also on the choice of initial term z_0. We often represent such iteration sequences, as in Figure 1.1, by plotting the points $z_0, z_1, \ldots$, and indicating how these points are related by the function f. Note that, in effect, we are plotting the domain and codomain of f on the same diagram!

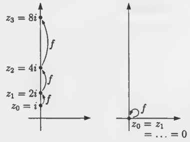

Figure 1.1

Problem 1.1

Calculate and plot the terms up to z_3 of each of the following iteration sequences. Also, write down the corresponding functions f.

(a) $z_{n+1} = z_n^2, \quad z_0 = i$ (b) $z_{n+1} = \frac{1}{2}z_n + 1, \quad z_0 = 0$

(c) $z_{n+1} = z_n^2 - 1, \quad z_0 = 0$ (d) $z_{n+1} = z_n^2 + i, \quad z_0 = 0$

In order to study iteration sequences systematically, it is useful to introduce various basic notations. For example, if the sequence $\{z_n\}$ is defined by

$$z_{n+1} = f(z_n), \qquad n = 0, 1, 2, \ldots,$$

then

$$z_1 = f(z_0), \quad z_2 = f(z_1) = f(f(z_0)), \quad z_3 = f(z_2) = f(f(f(z_0))),$$

and, in general,

$$z_n = f(f(\cdots(f(z_0))\cdots)), \qquad \text{for } n = 1, 2, \ldots, \tag{1.1}$$

where the function f is applied n times. We introduce a notation for such repeated compositions.

Definition The **nth iterate** of a function f is the function obtained by applying the function f exactly n times:

$$f^n = f \circ f \circ \cdots \circ f.$$

Also, f^0 denotes the identity function $f^0(z) = z$.

For example,
$$f^1(z) = f(z)$$
and
$$f^2(z) = f(f(z)).$$

Remarks

1 There is, of course, a possible confusion between

$$f^2 = f \circ f \qquad \text{and} \qquad f^2 = f \times f,$$

but the context will make the intended meaning clear.

There should be no confusion with the nth derivative $f^{(n)}$.

2 Equation (1.1), relating the general term z_n to the initial term z_0, can now be written in the convenient form

$$z_n = f^n(z_0), \qquad n = 1, 2, \ldots;$$

see Figure 1.2.

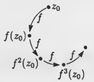

Figure 1.2

3 Note that if $m, n \geq 0$, then

$$f^m(f^n(z)) = f^{m+n}(z) = f^n(f^m(z)),$$

since composition of functions is associative.

Example 1.1

Determine the rules for the functions f^2 and f^3 when $f(z) = z^2 - 1$.

Solution

By the definition,

$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2, \end{aligned}$$

and so

$$\begin{aligned} f^3(z) &= f(f^2(z)) \\ &= (z^4 - 2z^2)^2 - 1 \\ &= z^8 - 4z^6 + 4z^4 - 1. \quad \blacksquare \end{aligned}$$

Alternatively,
$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= (z^2 - 1)^4 - 2(z^2 - 1)^2 \\ &= z^8 - 4z^6 + 4z^4 - 1. \end{aligned}$$

Problem 1.2

Determine the rules for the functions f^2 and f^3 when $f(z) = \frac{1}{2}z + 1$.

The solution to Example 1.1 suggests that for some functions f it may be difficult to find a general formula for the nth iterate, f^n. However, there are a few simple cases which are very useful.

Example 1.2

Find a formula for the nth iterate f^n of each of the following functions.

(a) $f(z) = az$, where $a \in \mathbb{C}$ (b) $f(z) = z^2$

Solution

(a) By definition,

$$f^1(z) = f(z) = az,$$
$$f^2(z) = f(f(z)) = f(az) = a(az) = a^2 z,$$
$$f^3(z) = f\left(f^2(z)\right) = f\left(a^2 z\right) = a\left(a^2 z\right) = a^3 z,$$

and, in general,

$$f^n(z) = a^n z, \qquad \text{for } n = 1, 2, \ldots .$$

This can be proved by Mathematical Induction.

(b) By definition,

$$f^1(z) = f(z) = z^2,$$
$$f^2(z) = f(f(z)) = f\left(z^2\right) = \left(z^2\right)^2 = z^4,$$
$$f^3(z) = f\left(f^2(z)\right) = f\left(z^4\right) = \left(z^4\right)^2 = z^8,$$

and, in general,

$$f^n(z) = z^{2^n}, \qquad \text{for } n = 1, 2, \ldots . \quad \blacksquare$$

The following problem deals with other nth iterates which can be found explicitly.

Problem 1.3

Find a formula for the nth iterate f^n of each of the following functions.

(a) $f(z) = z + b$, where $b \in \mathbb{C}$

(b) $f(z) = z^3$

The formulas obtained in Example 1.2 and Problem 1.3 can be used to determine the behaviour of the corresponding iteration sequences, as we now illustrate.

Example 1.3

(a) Prove that if $f(z) = az$, where $|a| < 1$, and $z_0 \in \mathbb{C}$, then

$$z_n = f^n(z_0) \to 0 \text{ as } n \to \infty.$$

(b) Prove that if $f(z) = z^2$ and $|z_0| < 1$, then

$$z_n = f^n(z_0) \to 0 \text{ as } n \to \infty.$$

(c) Prove that if $f(z) = z + b$, where $b \neq 0$, and $z_0 \in \mathbb{C}$, then

$$z_n = f^n(z_0) \to \infty \text{ as } n \to \infty.$$

Solution

(a) By Example 1.2(a),

$$z_n = f^n(z_0) = a^n z_0, \qquad \text{for } n = 1, 2, \ldots .$$

Since $|a| < 1$, we deduce that $\{z_n\}$ is a null sequence, as required.

Unit A3, Theorems 1.2 and 1.3

(b) By Example 1.2(b),

$$z_n = f^n(z_0) = z_0^{2^n}, \qquad \text{for } n = 1, 2, \ldots .$$

Now we use the facts that $|z_0| < 1$ and $2^n \geq n$, for $n = 1, 2, \ldots$, to deduce that

By the Binomial Theorem,
$2^n = (1+1)^n$
$= 1 + n + \cdots \geq n.$

$$|z_n| = |z_0|^{2^n} \leq |z_0|^n, \qquad \text{for } n = 1, 2, \ldots .$$

Since $|z_0| < 1$, $\{|z_0|^n\}$ is a null sequence and so, therefore, is $\{z_n\}$, by the Squeeze Rule.

Unit A3, Theorem 1.1

(c) By Problem 1.3(a),

$$z_n = f^n(z_0) = z_0 + nb, \qquad \text{for } n = 1, 2, \ldots .$$

Thus

$$\begin{aligned}\frac{1}{z_n} &= \frac{1}{z_0 + nb} \\ &= \frac{1/n}{z_0/n + b} \\ &\to \frac{0}{b} = 0 \text{ as } n \to \infty \text{ (since } b \neq 0);\end{aligned}$$

so, by the Reciprocal Rule,

Unit A3, Theorem 1.5

$$z_n \to \infty \quad \text{as } n \to \infty. \quad \blacksquare$$

Remark The sequence $\{z_0^{2^n}\}$ in part (b) tends to 0 extremely quickly if $|z_0| < 1$. In fact, for any α such that $0 < |\alpha| < 1$, we have

$$\frac{|z_0|^{2^n}}{\alpha^n} \to 0 \text{ as } n \to \infty.$$

Thus $\{z_0^{2^n}\}$ tends to 0 faster than $\{\alpha^n\}$ for any α with $0 < |\alpha| < 1$.

To prove this, note that if

$$a_n = |z_0|^{2^n}/|\alpha|^n, \quad n = 0, 1, 2, \ldots,$$

then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{|z_0|^{2^n}}{|\alpha|} \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $\{a_n\}$ is null, by *Unit B3*, Theorems 1.9 and 1.7, for example.

In Example 1.3(a) we saw that $f^n(z_0) \to 0$ as $n \to \infty$ for all choices of initial term z_0, whereas in Example 1.3(b), $f^n(z_0) \to 0$ as $n \to \infty$, whenever $|z_0| < 1$. In the next problem we ask you to investigate what happens for this latter function f with other initial values.

Problem 1.4

With $f(z) = z^2$, determine the behaviour of the iteration sequence

$$z_n = f^n(z_0), \qquad n = 1, 2, \ldots,$$

when

(a) $z_0 = 1$; (b) $z_0 = -i$; (c) $z_0 = e^{2\pi i/3}$; (d) $|z_0| > 1$.

1.2 Fixed points

Whenever an iteration sequence, defined by a *continuous* function f, converges to a limit α, say, then the point α has the property that $f(\alpha) = \alpha$, as we now show. Suppose that the iteration sequence $z_n = f^n(z_0)$ is such that

$$z_n \to \alpha \text{ as } n \to \infty.$$

Then $z_{n+1} \to \alpha$ as $n \to \infty$, so that

$$\alpha = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} f(z_n) = f(\alpha),$$

because the function f is continuous at α. As the limit α satisfies $f(\alpha) = \alpha$, it is called a *fixed point* of the function f (see Figure 1.3).

The sequence $\{z_{n+1}\}$ is just the sequence $\{z_n\}$ with its first term removed.

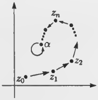

Figure 1.3

Definition A point α is a **fixed point** of a function f if $f(\alpha) = \alpha$.

For example, the function $f(z) = 2z$ has 0 as a fixed point, whereas the function $f(z) = z^2$ has 0 and 1 as fixed points. In general, we find the fixed points (if any) of a given function f by solving the **fixed point equation** $f(z) = z$.

Problem 1.5

Determine the fixed points of each of the following functions f.

(a) $f(z) = \frac{1}{2}z + 1$ (b) $f(z) = z^2 - 2$ (c) $f(z) = z^3$

The behaviour of an iteration sequence $z_n = f^n(z_0), n = 1, 2, \ldots$, near a fixed point α of an *analytic* function f depends to a very great extent on the derivative of f at α. If $|f'(\alpha)| < 1$, then, to a good approximation, f maps small discs with centre α to even smaller discs with centre α. Thus an initial term z_0 near α gives rise to an iteration sequence which is attracted to α (see Figure 1.4).

Recall from *Unit A4*, Subsection 1.5, that $f'(\alpha)$ acts as a complex scale factor at α.

Theorem 1.1 Let α be a fixed point of an analytic function f and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that

$$\lim_{n\to\infty} f^n(z_0) = \alpha, \qquad \text{for } |z_0 - \alpha| < r.$$

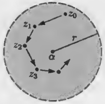

Figure 1.4

Proof We first choose a real number a such that $|f'(\alpha)| < a < 1$. Since

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \text{and} \quad a - |f'(\alpha)| > 0,$$

there is a positive number r such that

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) \right| < a - |f'(\alpha)|, \qquad \text{for } 0 < |z - \alpha| < r;$$

see Figure 1.5.

Apply the ε-δ definition of limit (*Unit A3*, Section 3) with

$$\varepsilon = a - |f'(\alpha)|$$

and put $r = \delta$.

Hence

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right| < a, \qquad \text{for } 0 < |z - \alpha| < r,$$

and so, since $f(\alpha) = \alpha$,

$$|f(z) - \alpha| \le a|z - \alpha|, \qquad \text{for } |z - \alpha| < r.$$

Thus if $|z_0 - \alpha| < r$, then

$$|f(z_0) - \alpha| \le a|z_0 - \alpha|,$$

so that $|f(z_0) - \alpha| < r$, also.

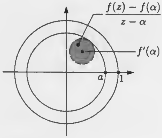

Figure 1.5

Hence

$$|f^2(z_0) - \alpha| \le a|f(z_0) - \alpha| \le a^2|z_0 - \alpha|,$$

and, in general,

$$|f^n(z_0) - \alpha| \le a^n|z_0 - \alpha|, \qquad \text{for } n = 1, 2, \ldots . \tag{1.2}$$

Since $0 < a < 1$, the sequence $\{a^n\}$ is a null sequence. Thus, for $|z_0 - \alpha| < r$, the sequence $\{f^n(z_0) - \alpha\}$ is null, by the Squeeze Rule, and so

$$\lim_{n\to\infty} f^n(z_0) = \alpha. \quad \blacksquare$$

In this proof, notice that the smaller $|f'(\alpha)|$ is, the smaller we can choose the number a (such that $|f'(\alpha)| < a < 1$) and so, by Inequality (1.2), the faster the sequence $\{f^n(z_0)\}$ converges to α. This convergence is very fast if $f'(\alpha) = 0$.

If α is a fixed point of f for which $|f'(\alpha)| > 1$, then we should expect initial terms z_0 near α (but not at α) to be pushed away from α by f. If $|f'(\alpha)| = 1$, then the behaviour depends in a more subtle way on the precise value of $f'(\alpha)$. These observations suggest the following classification of fixed points.

Definitions The fixed point α of an analytic function f is

(a) **attracting**, if $|f'(\alpha)| < 1$;

(b) **repelling**, if $|f'(\alpha)| > 1$;

(c) **indifferent**, if $|f'(\alpha)| = 1$;

(d) **super-attracting**, if $f'(\alpha) = 0$.

Some texts use
(a) *attractive* or *stable*;
(b) *repulsive* or *unstable*;
(c) *neutral*.

Note that (d) is a special case of (a).

For example, the function $f(z) = az$, where $a \in \mathbb{C}$, has 0 as a fixed point and, since $f'(z) = a$, this fixed point is attracting if $|a| < 1$, repelling if $|a| > 1$ and indifferent if $|a| = 1$.

Problem 1.6

For each of the following functions f, classify the given fixed point α.

(a) $f(z) = z^2, \quad \alpha = 0, 1$

(b) $f(z) = \frac{1}{2}z + 1, \quad \alpha = 2$

(c) $f(z) = z^2 - 2, \quad \alpha = 2$

For any given function f with an attracting fixed point α, it is natural to ask exactly which points z are attracted to α under iteration of f (that is, $f^n(z) \to \alpha$ as $n \to \infty$) and so we make the following definition.

Definition If α is an attracting fixed point of an analytic function f, then the **basin of attraction** of α under f is the set

$$\{z : f^n(z) \to \alpha \text{ as } n \to \infty\}.$$

Here, and subsequently, we may use z, rather than z_0, as an initial term, when we do not need to label the sequence $\{z_n\}$.

A simple example is the basin of attraction of $\alpha = 0$ under the function $f(z) = z^2$. This is the open unit disc because

$$f^n(z_0) \to 0 \text{ as } n \to \infty, \qquad \text{for } |z_0| < 1,$$

Recall from Example 1.3(b) that
$f^n(z_0) = z_0^{2^n}, \quad \text{for } n = 0, 1, 2, \ldots .$

but

$$f^n(z_0) \not\to 0 \text{ as } n \to \infty, \qquad \text{for } |z_0| \ge 1 \quad (\text{since } |f^n(z_0)| \ge 1, \text{ for } n = 1, 2, \ldots).$$

Later in the unit we see some more complicated examples.

Problem 1.7

For each of the following functions f, determine the basin of attraction of the given fixed point α.

(a) $f(z) = \frac{1}{2}z$, $\alpha = 0$ (b) $f(z) = z^3$, $\alpha = 0$.

1.3 Conjugate iteration sequences

Consider the iteration sequence

$$z_{n+1} = z_n^2 + 2z_n, \qquad n = 0, 1, 2, \ldots, \text{ with } z_0 = -\tfrac{1}{2}, \tag{1.3}$$

Here $z_n = f^n(z_0)$, where $f(z) = z^2 + 2z$.

and suppose that we wish to find a formula for z_n in terms of n. This iteration sequence is quite complicated, and so it is sensible to look at the terms of $\{z_n\}$:

$$z_0 = -\tfrac{1}{2}, \quad z_1 = -\tfrac{3}{4}, \quad z_2 = -\tfrac{15}{16}, \quad z_3 = -\tfrac{255}{256}, \ldots .$$

These terms suggest that $z_n \to -1$ as $n \to \infty$, and so we make the substitution

$$z_n = w_n - 1,$$

Thus $w_n = h(z_n)$, where $w = h(z) = z + 1$.

and try to find a formula for w_n. Substituting for z_n and z_{n+1} in (1.3), we obtain

$$\begin{aligned} w_{n+1} - 1 &= (w_n - 1)^2 + 2(w_n - 1) \\ &= w_n^2 - 2w_n + 1 + 2w_n - 2 \\ &= w_n^2 - 1; \end{aligned}$$

hence

$$w_{n+1} = w_n^2.$$

The iteration sequence

$$w_{n+1} = w_n^2, \qquad n = 0, 1, 2, \ldots, \text{ with } w_0 = z_0 + 1 = \tfrac{1}{2}, \tag{1.4}$$

Here $w_{n+1} = g(w_n)$, where $g(w) = w^2$.

is simpler than one given in (1.3) and, moreover, we know that

$$w_n = \left(\tfrac{1}{2}\right)^{2^n}, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Example 1.3(b)

We deduce that the formula for z_n in terms of n is

$$z_n = \left(\tfrac{1}{2}\right)^{2^n} - 1, \qquad \text{for } n = 0, 1, 2, \ldots .$$

More generally, suppose that

$$z_{n+1} = f(z_n), \qquad n = 0, 1, 2, \ldots,$$

is a given, but complicated, iteration sequence that we wish to investigate, and that h is a one-one function. Then

$$w_n = h(z_n), \qquad n = 0, 1, 2, \ldots,$$

is also an iteration sequence. Indeed, since $z_n = h^{-1}(w_n)$, we have

The inverse function h^{-1} exists because h is one-one.

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(f(z_n)) \\ &= h\left(f\left(h^{-1}(w_n)\right)\right), \qquad \text{for } n = 0, 1, 2, \ldots . \end{aligned}$$

Thus

$$w_{n+1} = g(w_n), \qquad \text{for } n = 0, 1, 2, \ldots,$$

where the function g is given by $g = h \circ f \circ h^{-1}$ (see Figure 1.6). With a suitable choice of h, the function g will be simpler than f (as in the above example, where $f(z) = z^2 + 2z$, $g(w) = w^2$ and $h(z) = z + 1$).

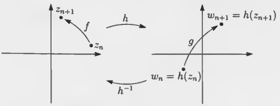

Figure 1.6 $g(w_n) = \left(h \circ f \circ h^{-1}\right)(w_n)$

We make the following definition.

Definition The functions f and g are **conjugate** to each other if

$$g = h \circ f \circ h^{-1},$$

for some one-one function h called the **conjugating function**. If the sequence $\{z_n\}$ is defined by

$$z_{n+1} = f(z_n), \qquad n = 0, 1, 2, \ldots,$$

for some z_0, and $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, then the sequence $\{w_n\}$ satisfies

$$w_{n+1} = g(w_n), \qquad \text{for } n = 0, 1, 2, \ldots,$$

and $\{z_n\}$ and $\{w_n\}$ are called **conjugate iteration sequences.**

Do not confuse this use of the word 'conjugate', which is borrowed from group theory, with 'complex conjugate'.

Remark In practice, the function g is usually found by substituting $z_n = h^{-1}(w_n)$ and $z_{n+1} = h^{-1}(w_{n+1})$ into $z_{n+1} = f(z_n)$ and then rearranging, as we did with (1.3).

Since the sequence $\{w_n\}$ is the image of the sequence $\{z_n\}$ under the function h, it follows that both sequences have the same behaviour. For example, if both h and h^{-1} are continuous functions, then $\{z_n\}$ is convergent if and only if $\{w_n\}$ is convergent.

Substituting $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, where h is a one-one function, amounts to making a change of variable.

Also, α is a fixed point of f if and only if $h(\alpha)$ is a fixed point of g.

If the conjugating function h is required to be one-one and entire, then it must in fact be of the form $h(z) = az + b$, where $a \neq 0$. (This can be proved using the Casorati–Weierstrass Theorem.) When dealing with the iteration of rational functions (as in the next subsection), we also use extended Möbius transformations as conjugating functions, since these are one-one on $\widehat{\mathbb{C}}$.

Problem 1.8

Prove that the iteration sequence

$$z_{n+1} = z_n - z_n^2, \qquad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + \tfrac{1}{4}, \qquad n = 0, 1, 2, \ldots,$$

with conjugating function $h(z) = -z + \frac{1}{2}$. If $z_0 = \frac{1}{2}$, what is w_0?

Problem 1.9

(a) Prove that the iteration sequence

$$z_{n+1} = az_n + b, \qquad n = 0, 1, 2, \ldots,$$

where $a \neq 1$, is conjugate to the iteration sequence

$$w_{n+1} = aw_n, \qquad n = 0, 1, 2, \ldots,$$

with conjugating function $h(z) = z + b/(a-1)$.

(b) Hence obtain a formula for z_n in terms of z_0 and describe the behaviour of the sequence $\{z_n\}$ when

(i) $|a| < 1$; (ii) $|a| = 1, a \neq 1$; (iii) $|a| > 1$.

You dealt with the case $a = 1$ in Problem 1.3(a).

1.4 The Newton–Raphson method

We shall now see how the ideas introduced so far help with the Newton–Raphson method, described in the Introduction. If p is a polynomial function, then the corresponding Newton–Raphson iteration sequence is

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \qquad n = 0, 1, 2, \ldots.$$

We call the function to be iterated here the **Newton–Raphson function** corresponding to p, and denote it by N; thus

$$N(z) = z - \frac{p(z)}{p'(z)}.$$

In general, N is a rational function (unless p is of degree 1) and has extension $\widehat{N}$ to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Thus a Newton–Raphson iteration sequence may include the point at infinity among its terms.

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A key observation is that if α is a simple zero of p, then $p'(\alpha) \neq 0$ and $N(\alpha) = \alpha$. Thus α is a fixed point of N. To classify it, we evaluate $N'(\alpha)$:

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$$\begin{aligned} N'(\alpha) &= 1 - \frac{(p'(\alpha))^2 - p(\alpha)p''(\alpha)}{(p'(\alpha))^2} \\ &= 1 - \frac{(p'(\alpha))^2}{(p'(\alpha))^2} \qquad \text{(since } p(\alpha) = 0\text{)} \\ &= 0. \end{aligned}$$

Thus a simple zero α of p is a super-attracting fixed point for the Newton–Raphson function N. This is good news because it means that the Newton–Raphson method always converges rapidly to a simple zero α of p provided that our initial guess is close enough to α.

If α is a zero of order k, where $k > 1$, then it can be shown that α is a fixed point of N such that

$$N'(\alpha) = 1 - 1/k < 1,$$

so the Newton–Raphson method still works, but not so well. See, for example, Problem 1.11.

In 1879, Cayley analysed the Newton–Raphson method when p is the quadratic function

$$p(z) = z^2 + az + b, \qquad \text{where } a, b \in \mathbb{C}.$$

Arthur Cayley (1821–1895) is well known for pioneering work in group theory and linear algebra.

If the function p has distinct zeros at α and β, say, then these zeros must be simple. Thus α and β are super-attracting fixed points of the Newton–Raphson function N. By Theorem 1.1, there are open discs around α and β whose points are attracted to α and β, respectively, under iteration of N. Cayley wished to know which points in $\mathbb{C}$ are attracted to α under iteration of N, and which are attracted to β. In other words, what are the basins of attraction of α and β under N?

Cayley found that the answer is remarkably simple: the perpendicular bisector of the line segment joining α and β forms a 'watershed' for this iteration process (see Figure 1.7). If z_0 falls on the same side of the watershed as α, then $N^n(z_0) \to \alpha$ as $n \to \infty$, but if z_0 falls on the other side, then $N^n(z_0) \to \beta$ as $n \to \infty$. If z_0 falls exactly on the watershed line, then the sequence $\{N^n(z_0)\}$ remains on the line!

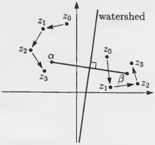


Figure 1.7

If we consider the Newton–Raphson function for $p(z) = z^2 + az + b$,

$$\begin{aligned} N(z) &= z - \frac{p(z)}{p'(z)} \\ &= z - \frac{z^2 + az + b}{2z + a} \\ &= \frac{z^2 - b}{2z + a}, \end{aligned}$$

then it is not at all evident why Cayley's result should be true. However, we obtain a much simpler iteration sequence by using the conjugating function

$$h(z) = \frac{z - \alpha}{z - \beta},$$

which is a Möbius transformation. The extended Möbius transformation $\widehat{h}$ is one-one on $\widehat{\mathbb{C}}$ and maps α to 0 and β to ∞. Also h has the remarkable property

Unit D1, Section 2

that

$$h(N(z)) = (h(z))^2, \qquad (1.5)$$

Identity (1.5) may be written in the form
$N = h^{-1} \circ g \circ h,$
where $g(w) = w^2$.

which we ask you to prove in Problem 1.10.

Thus, if

$$z_{n+1} = N(z_n), \qquad n = 0, 1, 2, \ldots,$$

and if

$$w_n = h(z_n), \qquad \text{for } n = 0, 1, 2, \ldots,$$

then

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(N(z_n)) \\ &= (h(z_n))^2 \qquad \text{(by Equation (1.5))} \\ &= w_n^2, \qquad \text{for } n = 0, 1, 2, \ldots . \end{aligned}$$

Thus

$$w_n = g^n(w_0), \qquad \text{for } n = 1, 2, \ldots,$$

where $g(w) = w^2$.

Now we know that

$$w_n \to 0 \text{ as } n \to \infty, \qquad \text{if } |w_0| < 1, \qquad (1.6)$$

Example 1.3(b) and Problem 1.4(d)

that

$$w_n \to \infty \text{ as } n \to \infty, \qquad \text{if } |w_0| > 1, \qquad (1.7)$$

and also that

the sequence $\{w_n\}$ remains on the circle $\{w : |w| = 1\}$ if $|w_0| = 1$.

To find out what happens to the original sequence $\{z_n\}$, we note that

$$h^{-1}(w_n) = z_n, \quad \widehat{h}^{-1}(0) = \alpha, \quad \widehat{h}^{-1}(\infty) = \beta$$

and

$$|w_0| = |h(z_0)| = \left| \frac{z_0 - \alpha}{z_0 - \beta} \right| .$$

Therefore we deduce from (1.6) and (1.7) that

Note that the function
$$h^{-1}(w) = \frac{-\beta w + \alpha}{-w + 1}$$
is analytic at 0 and has a removable singularity at ∞.

$$z_n \to \alpha \text{ as } n \to \infty, \qquad \text{if } |z_0 - \alpha| < |z_0 - \beta|,$$

that

$$z_n \to \beta \text{ as } n \to \infty, \qquad \text{if } |z_0 - \alpha| > |z_0 - \beta|,$$

and also that

$\{z_n\}$ remains on the extended line $\{z : |z - \alpha| = |z - \beta|\} \cup \{\infty\}$ if z_0 is on this line.

This is Cayley's remarkable solution.

Problem 1.10

Prove Identity (1.5).
(*Hint*: Note that both $z = \alpha$ and $z = \beta$ satisfy the equation $z^2 = -(az + b)$.)

Problem 1.11

Describe what happens under iteration of N if $\alpha = \beta$.

Finally, we look briefly at the Newton–Raphson method for cubic polynomial functions. Here, you might guess that the complex plane divides itself into three simple regions, each surrounding one zero of p and consisting of those points which are attracted to that zero under iteration of the Newton–Raphson function N. Such indeed seems to have been Cayley's hunch in 1879, although he was unable to prove such a result. With the help of computer-generated

pictures we can now see why Cayley had no chance of finding a simple solution to this problem!

Consider $p(z) = z^3 - 1$, whose zeros are

$$\alpha_1 = 1, \quad \alpha_2 = e^{2\pi i/3} = \tfrac{1}{2}(-1 + i\sqrt{3}), \quad \alpha_3 = e^{4\pi i/3} = \tfrac{1}{2}(-1 - i\sqrt{3}).$$

In this case

$$\begin{aligned} N(z) &= z - \frac{z^3 - 1}{3z^2} \\ &= \frac{2z^3 + 1}{3z^2} \end{aligned}$$

and, as before, α_1, α_2, α_3 are each super-attracting fixed points of N. Figure 1.8 shows in white the basin of attraction of the fixed point $\alpha_1 = 1$.

Figure 1.8 The basin of attraction of 1

The basins of attraction of α_1, α_2, α_3 must be congruent to each other under rotation about 0 through $2\pi/3$, because of the symmetry of α_1, α_2, α_3. But these basins are not at all simple, and they are not even regions (because they are, in fact, not connected). The union of these three strange basins is almost the whole of $\mathbb{C}$. In addition, there is a complicated 'watershed' which separates the basins of attraction (see Figure 1.9) and which manages, somehow, to be the boundary of all three sets simultaneously!

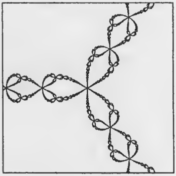


Figure 1.9 The watershed

Thus the iteration of even fairly simple rational functions can lead to very complicated behaviour. In the next section, we find that complicated behaviour can occur even for the iteration of simple polynomial functions.

We return to the Newton–Raphson method in Section 5.

2 ITERATING COMPLEX QUADRATICS

After working through this section, you should be able to:

(a) conjugate a given quadratic iteration sequence to one determined by a member of the family of functions $\{P_c : c \in \mathbb{C}\}$, where $P_c(z) = z^2 + c$;

(b) explain why $P_c^n(z_0) \to \infty$ as $n \to \infty$ if $|z_0|$ is large enough;

(c) describe some of the properties of the *escape set* E_c of P_c and of its complement, the *keep set* K_c;

(d) find the *periodic points* of P_c, for certain values of c, and find their nature;

(e) understand the definition of the *Julia set* J_c.

2.1 The basic quadratic family

In Section 1 we saw that iteration sequences of the form

$$z_{n+1} = az_n + b, \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

can be completely analysed; that is, for all a, $b \in \mathbb{C}$, we can describe the behaviour of $\{z_n\}$ for any initial term z_0. In this section we begin to study iteration sequences of the form

Example 1.3(c) and Problem 1.9

$$z_{n+1} = az_n^2 + bz_n + c, \qquad n = 0, 1, 2, \ldots, \tag{2.2}$$

where $a \neq 0$. We shall find that the family of such sequences is much more diverse than that given by (2.1). To begin with we note that every iteration sequence of the form (2.2) is in fact conjugate to one of a simpler type.

Theorem 2.1 The iteration sequence

$$z_{n+1} = az_n^2 + bz_n + c, \qquad n = 0, 1, 2, \ldots,$$

where $a \neq 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \qquad n = 0, 1, 2, \ldots, \tag{2.3}$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is

$$h(z) = az + \tfrac{1}{2}b.$$

Problem 1.8 is a special case of this result, with $a = -1$, $b = 1$, $c = 0$.

Proof The recurrence relation (2.2) can be rearranged as

$$az_{n+1} = \left(az_n + \tfrac{1}{2}b\right)^2 + ac - \tfrac{1}{4}b^2, \qquad n = 0, 1, 2, \ldots.$$

Multiply by a and complete the square.

Thus, putting $w_n = h(z_n)$, for $n = 0, 1, 2, \ldots$, where $h(z) = az + \frac{1}{2}b$, we obtain

$$w_{n+1} - \tfrac{1}{2}b = w_n^2 + ac - \tfrac{1}{4}b^2, \quad n = 0, 1, 2, \ldots;$$

that is,

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$, as required. ■

There are many different iteration sequences of the form (2.2) which are conjugate to any one iteration sequence of the form (2.3). This is illustrated in the following problem.

Problem 2.1

Use Theorem 2.1 to show that each of the following iteration sequences

(a) $z_{n+1} = 4z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots$, with $z_0 = \frac{1}{2}$,

(b) $z_{n+1} = 1 - 2z_n^2, \qquad n = 0, 1, 2, \ldots$, with $z_0 = 0$,

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - 2, \qquad n = 0, 1, 2, \ldots, \text{ with } w_0 = 0.$$

Theorem 2.1 tells us that if we wish to understand the possible behaviour of quadratic functions under iteration, then it is sufficient to consider only those of the form (2.3), and it is convenient to relabel these as

$$z_{n+1} = z_n^2 + c, \qquad n = 0, 1, 2, \ldots,$$

where c is a complex parameter. We shall devote most of the rest of this unit to such iteration sequences, and so we introduce a name for the corresponding quadratic functions.

> **Definition** The set of functions $\{P_c : c \in \mathbb{C}\}$ defined by
>
> $$P_c(z) = z^2 + c,$$
>
> where $c \in \mathbb{C}$, is the **family of basic quadratic functions**.

For example, $P_0(z) = z^2$, $P_1(z) = z^2 + 1$ and $P_i(z) = z^2 + i$ are all basic quadratic functions. In the following problems we ask you to establish some elementary properties of the basic quadratic functions.

Problem 2.2

(a) Determine the rules for the functions P_c^2 and P_c^3.

(b) Write down a formula for $P_c^{n+1}(z)$ in terms of $P_c^n(z)$, and hence prove that P_c^n is an even polynomial function of degree 2^n.

Remember that P_c^n denotes the nth iterate of P_c.

Problem 2.3

(a) Show that the fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, and prove that at least one of these is repelling $\left(\text{unless } c = \frac{1}{4}\right)$.
(Hint: If α and β are the fixed points of P_c, then
$\frac{1}{2}(P_c'(\alpha) + P_c'(\beta)) = 1$.)

(b) What happens if $c = \frac{1}{4}$?

2.2 The escape set and the keep set

In Example 1.2(b) we observed that the iteration sequence

$$z_n = P_0^n(z_0), \qquad n = 0, 1, 2, \ldots,$$

can be determined explicitly as

$$z_n = z_0^{2^n}, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Here
$P_0(z) = z^2$,
so that $c = 0$.

Thus

$$\left.\begin{array}{ll} z_n \to 0 \text{ as } n \to \infty, & \text{if } |z_0| < 1, \\ z_n \to \infty \text{ as } n \to \infty, & \text{if } |z_0| > 1, \\ |z_n| = 1, \text{ for } n = 1, 2, \ldots, & \text{if } |z_0| = 1. \end{array}\right\} \qquad (2.4)$$

It is natural to ask whether a similar behaviour occurs for other values of c. We shall see later that when the initial term z_0 is small, the sequence $z_n = P_c^n(z_0)$, $n = 1, 2, \ldots$, behaves in dramatically different ways for different values of c. However, when the initial term z_0 is large these sequences behave in essentially the same way, for all values of c, as we now show.

Theorem 2.2 Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,

$$\{|P_c^n(z_0)|\} \text{ is an increasing sequence,}$$

and

$$P_c^n(z_0) \to \infty \text{ as } n \to \infty.$$

If c = 0, then $r_c = 1$ and we recover the middle line of (2.4).

Proof First note that, by the backwards form of the Triangle Inequality,

$$|P_c(z)| = |z^2 + c| \geq |z|^2 - |c|. \tag{2.5}$$

The number r_c is the positive solution of the equation $x^2 - |c| = x$ (see Figure 2.1), and we claim that if $\varepsilon > 0$, then

$$x^2 - |c| \geq (1+\varepsilon)x, \qquad \text{for } x \geq r_c + \varepsilon. \tag{2.6}$$

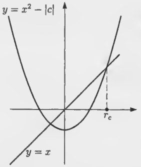


Figure 2.1 $r_c^2 - |c| = r_c$

Indeed, if $x \geq r_c + \varepsilon$, then

$$\begin{aligned}
\frac{x^2 - |c|}{x} &= x - \frac{|c|}{x} \\
&\geq (r_c + \varepsilon) - \frac{|c|}{r_c + \varepsilon} \\
&= \frac{r_c^2 + 2r_c\varepsilon + \varepsilon^2 - |c|}{r_c + \varepsilon} \\
&= \frac{r_c + 2r_c\varepsilon + \varepsilon^2}{r_c + \varepsilon} \qquad \left(\text{since } r_c^2 - |c| = r_c\right) \\
&\geq \frac{r_c + r_c\varepsilon + \varepsilon + \varepsilon^2}{r_c + \varepsilon} \qquad (\text{since } r_c \geq 1) \\
&= 1 + \varepsilon,
\end{aligned}$$

as required for Inequality (2.6). Inequalities (2.5) and (2.6) now give

$$|P_c(z)| \geq (1+\varepsilon)|z| > |z|, \qquad \text{for } |z| \geq r_c + \varepsilon.$$

If $|z_0| \geq r_c + \varepsilon$, then we can apply this inequality successively to z_0, $P_c(z_0)$, $P_c^2(z_0), \ldots$, to deduce that the sequence $\{|P_c^n(z_0)|\}$ is increasing, and

$$|P_c^n(z_0)| \geq (1+\varepsilon)^n |z_0|, \qquad \text{for } n = 1, 2, \ldots.$$

Hence $P_c^n(z_0) \to \infty$ as $n \to \infty$. Since ε is any positive number, the proof is complete. ■

Remarks

1 Theorem 2.2 may be easier to understand if you think of ∞ as a fixed point of the extended function $\widehat{P}_c$ of P_c. To discover the nature of this fixed point, we use the conjugating function $h(z) = 1/z$, which moves the fixed point from ∞ to 0. By this means the iteration sequence

See *Unit D1*, Subsection 2.2.

$$z_{n+1} = z_n^2 + c, \qquad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence $\{w_n\}$, where

$$\frac{1}{w_{n+1}} = \left(\frac{1}{w_n}\right)^2 + c, \qquad n = 0, 1, 2, \ldots;$$

$w_n = h(z_n) = \dfrac{1}{z_n} \implies z_n = \dfrac{1}{w_n}$

that is,

$$w_{n+1} = \frac{w_n^2}{1 + cw_n^2}, \qquad n = 0, 1, 2, \ldots.$$

As expected, because $\widehat{P}_c(\infty) = \infty$, the corresponding function $Q_c(w) = w^2/\left(1 + cw^2\right)$ has a fixed point at 0, which turns out to be super-attracting. Thus, by Theorem 1.1,

Since $Q_c'(w) = \dfrac{2w}{\left(1 + cw^2\right)^2}$, we have $Q_c'(0) = 0$.

$$w_n = Q_c^n(w_0) \to 0 \text{ as } n \to \infty, \text{ provided that } |w_0| \text{ is small enough,}$$

and so

$$z_n = P_c^n(z_0) \to \infty \text{ as } n \to \infty, \text{ provided that } |z_0| \text{ is large enough.}$$

2 The formula $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ will play a significant role later in the unit; therefore we indicate in Figure 2.2 the graph of the function $|c| \longmapsto r_c$, together with the graph of the identity function $|c| \longmapsto |c|$ for comparison. Notice, in particular, that

$$|c| \leq r_c \iff |c| \leq 2 \tag{2.7}$$

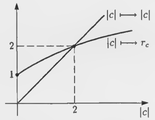


Figure 2.2

Problem 2.4

(a) Verify that

$$r_0 = 1, \quad r_i = \tfrac{1}{2}(1+\sqrt{5}) \quad \text{and} \quad r_{-2} = 2.$$

(b) Show that the sequences $\{P_0^n(1)\}$ and $\{P_{-2}^n(2)\}$ are constant. What does this tell you about the values r_0 and r_{-2} in relation to Theorem 2.2?

We now investigate the set of *all* points which are attracted to ∞ or 'escape' to ∞ under iteration of P_c. We call this set the *escape set*; its complement is the set of points which we 'keep' under iteration of P_c.

> **Definition** For $c \in \mathbb{C}$, the **escape set** E_c is
>
> $$E_c = \{z : P_c^n(z) \to \infty \text{ as } n \to \infty\}.$$
>
> The **keep set** K_c is the complement of E_c.

Remark The escape set E_c may be thought of as the basin of attraction of ∞.

For example, we know by (2.4) that

$$\begin{aligned} E_0 &= \left\{z : P_0^n(z) = z^{2^n} \to \infty \text{ as } n \to \infty\right\} \\ &= \{z : |z| > 1\}, \end{aligned}$$

and hence that

$$\begin{aligned} K_0 &= \mathbb{C} - E_0 \\ &= \{z : |z| \leq 1\} \end{aligned}$$

(see Figure 2.3).

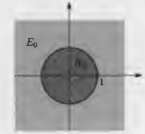
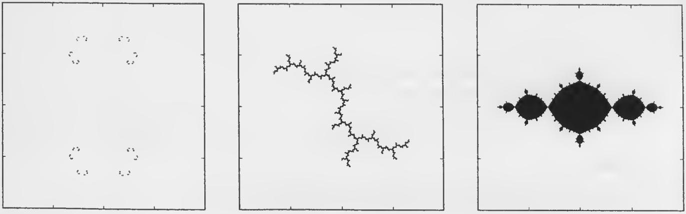


Figure 2.3

Here, and throughout this section, the square represented in such figures is

$$\{z : -2 \leq \operatorname{Re} z, \operatorname{Im} z \leq 2\}.$$

This example is rather misleading as, for most values of c, E_c does not have a simple shape. Figure 2.4 shows several examples of sets E_c (in white!) and K_c, plotted by computer.



(a) E_c and $K_c : c = 1$ (b) E_c and $K_c : c = i$ (c) E_c and $K_c : c = -1$

Figure 2.4

In fact, $c = -2$ is the only other value for which E_c and K_c have a simple shape. We now ask you to investigate this case.

Problem 2.5

Let L be the line segment $\{x + iy : |x| \leq 2, y = 0\}$.

L is the interval $[-2, 2]$.

(a) Let

$$z_{n+1} = z_n^2 - 2, \qquad n = 0, 1, 2, \ldots .$$

Prove that

if $z_0 \in L$, then $z_n \in L$, for $n = 1, 2, \ldots$;
if $z_0 \in \mathbb{C} - L$, then $z_n \in \mathbb{C} - L$, for $n = 1, 2, \ldots$.

(b) Prove that if $z_0 \in \mathbb{C} - L$, then the sequence $\{z_n\}$ in part (a) is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2, \qquad n = 0, 1, 2, \ldots,$$

and deduce that $z_n \to \infty$ as $n \to \infty$.
(*Hint*: Recall from *Unit D1*, Subsection 4.4, that the Joukowski function $J(w) = w + 1/w$ maps $\{w : |w| > 1\}$ one-one and conformally onto $\mathbb{C} - L$, put $w_n = J^{-1}(z_n)$, for $n = 0, 1, 2, \ldots$, and verify that $J(w_{n+1}) = J(w_n^2)$.)

(c) Deduce that $E_{-2} = \mathbb{C} - L$ and $K_{-2} = L$ (see Figure 2.5).

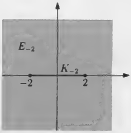


Figure 2.5

Though the set E_c is usually very complicated, a number of general observations can be made about it. For example, we can show that E_c has the property of being *completely invariant* under P_c.

> **Definition** A set A is **completely invariant** under a function f if
>
> $z \in A \iff f(z) \in A.$

This means that if z lies in A, then $f(z)$ and all its iterates also lie in A and, moreover, any point whose iterates eventually lie in A must itself lie in A. For example, it is easy to check that each of the sets $\mathbb{C}, \{0\}, \mathbb{C} - \{0\}, \{z : |z| < 1\}$, $\{z : |z| = 1\}$ and $\{z : |z| > 1\}$ is completely invariant under the function $f(z) = z^2$.

The following result lists several key facts about E_c and K_c.

> **Theorem 2.3** For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:
>
> (a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \leq r_c\}$;
>
> (b) E_c is open and K_c is closed;
>
> (c) $E_c \neq \mathbb{C}$ and $K_c \neq \varnothing$;
>
> (d) E_c and K_c are each completely invariant under P_c;
>
> (e) E_c and K_c are each symmetric under rotation by π about 0;
>
> (f) E_c is connected and K_c has no holes in it.

The statement that K_c has no holes in it just means that E_c is connected.

Remarks

1 In each case the property of K_c is equivalent to the corresponding property of E_c. Thus in the proof we need only establish the results for E_c.

2 Note that Properties (a), (b) and (c) combine to tell us that K_c is a non-empty compact set.

Proof of Theorem 2.3

This proof may be omitted on a first reading.

(a) We have already proved part (a) in Theorem 2.2.

(b) Suppose that $z_0 \in E_c$. Then, by definition, $P_c^n(z_0) \to \infty$ as $n \to \infty$. Thus, for some n_0, $|P_c^{n_0}(z_0)| > r_c$.

Let $\varepsilon = |P_c^{n_0}(z_0)| - r_c$. Since $\varepsilon > 0$ and $P_c^{n_0}$ is a polynomial function, $P_c^{n_0}$ is continuous at z_0 and so there exists $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Longrightarrow \quad |P_c^{n_0}(z) - P_c^{n_0}(z_0)| < \varepsilon$$

and hence

$$|z - z_0| < \delta \quad \Longrightarrow \quad |P_c^{n_0}(z)| > r_c$$

(see Figure 2.6).

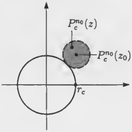


Figure 2.6

It follows from Theorem 2.2 that

$$|z - z_0| < \delta \quad \Longrightarrow \quad P_c^n(z) \to \infty \text{ as } n \to \infty,$$

so that $\{z : |z - z_0| < \delta\} \subseteq E_c$. This shows that E_c is open.

(c) The set E_c is not the whole of $\mathbb{C}$, because it does not contain the fixed points of P_c, found in Problem 2.3. (These must lie in K_c.)

(d) We ask you to prove part (d) in Problem 2.6.

(e) Since

$$P_c^n(-z) = P_c^n(z), \qquad \text{for } n = 1, 2, \ldots,$$

because P_c^n is an even function (see Problem 2.2(b)), we have

$$P_c^n(-z) \to \infty \Longleftrightarrow P_c^n(z) \to \infty$$

and so

$$-z \in E_c \Longleftrightarrow z \in E_c.$$

(f) Finally, we show that E_c is connected, that is, each pair of points in E_c can be joined by a path lying in E_c. Since the set $A_c = \{z : |z| > r_c\}$ is connected and $A_c \subseteq E_c$, it is sufficient to show that each point α in E_c can be joined to some point of A_c by a path in E_c. We prove this by contradiction.

Suppose in fact that α is a point of E_c which *cannot* be joined to the set A_c in this way. We define the following set:

$$\mathcal{R} = \{z \in E_c : z \text{ can be joined to } \alpha \text{ by a path in } E_c\}$$

(see Figure 2.7).

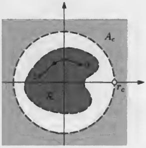


Figure 2.7

Then $\mathcal{R} \neq \varnothing$ (since $\alpha \in \mathcal{R}$), $\mathcal{R}$ is open (because if z can be joined to α in E_c, then so can points of any open disc in E_c with centre z) and $\mathcal{R}$ is connected (because pairs of points in $\mathcal{R}$ can be joined in $\mathcal{R}$ via α). Thus $\mathcal{R}$ is a subregion of E_c. Since α cannot be joined in $\mathcal{R}$ to A_c and $\mathcal{R}$ is open, we deduce that $\mathcal{R} \subseteq \{z : |z| < r_c\}$.

We can now use the Maximum Principle. If $\beta \in \partial\mathcal{R}$, then β does not lie in E_c (since, otherwise, we could enlarge $\mathcal{R}$ slightly). Thus

Unit C2, Theorem 4.2

$$|P_c^n(\beta)| \leq r_c, \qquad \text{for } n = 1, 2, \ldots.$$

By applying the Maximum Principle to each of the polynomial functions P_c^n on $\mathcal{R}$, we obtain

$$|P_c^n(z)| \leq r_c, \qquad \text{for } n = 1, 2, \ldots, \text{ and } z \in \mathcal{R},$$

which contradicts the fact that $\mathcal{R} \subseteq E_c$. Hence E_c is connected. ■

Problem 2.6

Prove part (d) of Theorem 2.3.

2.3 Periodic points

Figure 2.4 shows that the keep sets K_c have a remarkably diverse form. In order to investigate the shape of K_c, we need to identify as many points in K_c as possible. We already know that the fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ of P_c lie in K_c, and we can find other points of K_c by generalizing the notion of a fixed point.

Problem 2.3(a)

As an example, consider the function $P_{-1}(z) = z^2 - 1$, whose two fixed points $\frac{1}{2}(1 \pm \sqrt{5})$ must lie in K_{-1}. In addition, the points 0 and -1, have the property that

$$P_{-1}(0) = -1 \quad \text{and} \quad P_{-1}(-1) = 0 \tag{2.8}$$

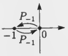


Figure 2.8

(see Figure 2.8). Thus the sequence $P_{-1}^n(0), n = 0, 1, 2, \ldots$, cycles endlessly between the points 0 and -1, as does the sequence $P_{-1}^n(-1)$, $n = 0, 1, 2, \ldots$. This means that 0 and -1 must both lie in K_{-1}. Since K_{-1} is symmetric under a rotation by π about 0, we can begin to build up a picture of K_{-1} (see Figure 2.9). This is very far from the complicated set in Figure 2.4(c), but at least it is a start!

Theorem 2.3(e)

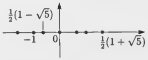


Figure 2.9

The fact that 0 and -1 satisfy Equations (2.8) can be interpreted as saying that 0 and -1 are both fixed points of the second iterate

$$\begin{aligned} P_{-1}^2(z) &= P_{-1}(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2. \end{aligned}$$

Indeed, it is evident that $P_{-1}^2(0) = 0$ and $P_{-1}^2(-1) = -1$.

Points of this type, which are fixed points of higher iterates of a function f, are called *periodic points* of f; they repeat periodically under iteration of f.

Definition The point α is a **periodic point**, with period p, of a function f if

$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \ldots, p-1.$$

The p points

$$\alpha,\ f(\alpha),\ f^2(\alpha),\ \ldots,\ f^{p-1}(\alpha)$$

then form a **cycle of period p**, or a **p-cycle** of f (see Figure 2.10).

Thus p is the smallest positive integer such that
$f^p(\alpha) = \alpha$.

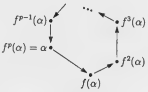


Figure 2.10

Remarks

1 Note that if we apply f repeatedly to points of a p-cycle, then we just obtain points of the p-cycle. Also the points of a p-cycle must be distinct. Indeed, if we had

$$f^k(\alpha) = f^\ell(\alpha), \qquad \text{where } 0 \le k < \ell \le p-1,$$

then the point $f^p(\alpha) = f^{p-k}(f^k(\alpha))$ would have to lie among the terms

$$f^{k+1}(\alpha), \ldots, f^\ell(\alpha),$$

which do not include α, which is a contradiction. Therefore, the points of a p-cycle are each distinct periodic points of f with period p.

2 Note that a fixed point of f is a 1-cycle of f.

3 Evidently, all the periodic points of P_c lie in the keep set K_c.

Determining the periodic points with period p of a given function f, for a given $p > 1$, is usually more difficult than determining the fixed points of f. This is because the equation

$$f^p(z) = z \tag{2.9}$$

usually has many more solutions as p increases and the rule for f^p is usually more complicated. Notice, however, that not all solutions of Equation (2.9) need be periodic points with period p. For example, any fixed point of f also satisfies Equation (2.9). More generally, if q is a factor of p, then any solution of $f^q(z) = z$ is also a solution of $f^p(z) = z$.

Example 2.1

Determine all periodic points with period 2 of the function $P_0(z) = z^2$, and write down the corresponding 2-cycles.

Solution

Since $P_0^2(z) = (z^2)^2 = z^4$, we have to solve the equation $z^4 = z$:

$$\begin{aligned} z^4 = z &\iff z^4 - z = 0 \\ &\iff z(z^3 - 1) = 0. \end{aligned}$$

The solutions of this quartic equation are $0, 1, e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}i)$ and $e^{4\pi i/3} = \frac{1}{2}(-1 - \sqrt{3}i)$. Of these, the points 0, 1 are fixed points of P_0, whereas

$$P_0\left(e^{2\pi i/3}\right) = \left(e^{2\pi i/3}\right)^2 = e^{4\pi i/3},$$

and

$$P_0\left(e^{4\pi i/3}\right) = \left(e^{4\pi i/3}\right)^2 = e^{2\pi i/3}.$$

Hence both $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are periodic points of P_0 with period 2, and they belong to the 2-cycle $e^{2\pi i/3}$, $e^{4\pi i/3}$. ■

Note that, as expected, the 2-cycle found in this example lies in the keep set $K_0 = \{z : |z| \leq 1\}$ (see Figure 2.11).

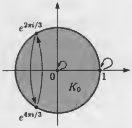


Figure 2.11

Problem 2.7

(a) Prove that $-i$ is a periodic point with period 2 of the function $P_i(z) = z^2 + i$. Hence find five points in K_i, none of them fixed points of P_i, and plot them.

The set K_i was plotted in Figure 2.4(b).

(b) Determine all periodic points with period 3 of the function $P_0(z) = z^2$, and write down the corresponding 3-cycles. Plot all the fixed points, 2-cycles and 3-cycles of P_0 on the same diagram.

(c) Prove that $\frac{1}{2}(-1 + \sqrt{2})$ is a periodic point of $P_{-5/4}$.

We now return to the function $P_{-1}(z) = z^2 - 1$, which has the 2-cycle 0, -1; that is, 0 and -1 are fixed points of the second iterate

$$P_{-1}^2(z) = z^4 - 2z^2.$$

Since

$$\begin{aligned} \left(P_{-1}^2\right)'(z) &= 4z^3 - 4z \\ &= 4z(z^2 - 1), \end{aligned}$$

it follows that

$$\left(P_{-1}^2\right)'(0) = 0 \quad \text{and} \quad \left(P_{-1}^2\right)'(-1) = 0,$$

so both 0 and -1 are super-attracting fixed points of P_{-1}^2. The fact that these two derivatives have the same value is no accident, as the following result shows.

Theorem 2.4 Let $\alpha,\ f(\alpha),\ f^2(\alpha),\dots,f^{p-1}(\alpha)$ form a p-cycle of an analytic function f. Then

(a) $$(f^p)'(\alpha) = f'(\alpha)\times f'(f(\alpha))\times f'(f^2(\alpha))\times\cdots\times f'(f^{p-1}(\alpha)) \quad (2.10)$$

and, moreover,

(b) the derivative of f^p takes the same value at each point of the p-cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \dots = (f^p)'(f^{p-1}(\alpha)).$$

Proof Since $f^p(z) = f(f(\cdots(f(z))\cdots))$, where the function f is applied p times, we deduce from repeated applications of the Chain Rule that

Unit A4, Theorem 3.1

$$(f^p)'(z) = f'(f^{p-1}(z))\times\cdots\times f'(f(z))\times f'(z).$$

By putting $z=\alpha$, we obtain (a).

Thus $(f^p)'(\alpha)$ is the product of the derivatives of f at the points of the p-cycle, and so $(f^p)'(f(\alpha))$ is also the product of the derivatives of f at the points of the p-cycle, and similarly for the other points $f^2(\alpha),\dots,f^{p-1}(\alpha)$. This establishes (b). ■

Theorem 2.4 allows us to classify the periodic points of an analytic function f, by using the number $(f^p)'(\alpha)$, which is called the **multiplier** of the corresponding cycle. We shall see shortly that different types of cycle lie in different parts of the keep set.

Some texts use the name *eigenvalue* rather than multiplier.

Definitions If α is a periodic point, with period p, of an analytic function f, then α and the corresponding p-cycle are

(a) **attracting**, if $|(f^p)'(\alpha)| < 1$;

(b) **repelling**, if $|(f^p)'(\alpha)| > 1$;

(c) **indifferent**, if $|(f^p)'(\alpha)| = 1$;

(d) **super-attracting**, if $(f^p)'(\alpha) = 0$.

In the next example we demonstrate two ways to calculate the multiplier.

Example 2.2

Determine the nature of the periodic point $e^{2\pi i/3}$ of $P_0(z)=z^2$.

Solution

In Example 2.1 we found that $e^{2\pi i/3}$ is a periodic point, with period 2, of the function P_0. Since $P_0^2(z)=z^4$, the multiplier is

$$\begin{aligned}(P_0^2)'\left(e^{2\pi i/3}\right) &= 4\left(e^{2\pi i/3}\right)^3 \quad (\text{since } (P_0^2)'(z)=4z^3)\\ &= 4,\end{aligned}$$

so that $\left|(P_0^2)'(e^{2\pi i/3})\right| > 1$. Thus $e^{2\pi i/3}$ is a repelling periodic point of P_0.

Alternatively, the point $e^{2\pi i/3}$ is part of the 2-cycle $e^{2\pi i/3}$, $e^{4\pi i/3}$ for the function P_0. Thus, by Theorem 2.4(a), the multiplier of the 2-cycle $e^{2\pi i/3}$, $e^{4\pi i/3}$ is

$$\begin{aligned}P_0'\left(e^{2\pi i/3}\right)P_0'\left(e^{4\pi i/3}\right) &= \left(2e^{2\pi i/3}\right)\left(2e^{4\pi i/3}\right) \quad (\text{since } P_0'(z)=2z),\\ &= 4,\end{aligned}$$

as before. ■

Note that the second method avoids the calculation of the rule for the pth iterate, which can be involved.

Problem 2.8

Determine the nature of each of the following periodic points, found in Problem 2.7.

(a) $-i$, a periodic point of P_i

(b) $e^{2\pi i/7}$, a periodic point of P_0

(c) $\frac{1}{2}(-1+\sqrt{2})$, a periodic point of $P_{-5/4}$

We now look more closely at where the periodic points of P_c lie in K_c. As an example, recall that the function $P_{-1}(z) = z^2 - 1$ has fixed points at $\frac{1}{2}(1 \pm \sqrt{5})$, which are both repelling because

$$\left|P'_{-1}\left(\tfrac{1}{2}(1+\sqrt{5})\right)\right| = 1+\sqrt{5} > 1 \quad \text{and} \quad \left|(P_{-1})'\left(\tfrac{1}{2}(1-\sqrt{5})\right)\right| = \sqrt{5}-1 > 1.$$

$P'_{-1}(z) = 2z$

Also, we saw before the proof of Theorem 2.4 that P_{-1} has the super-attracting 2-cycle $0, -1$. In Figure 2.12, these points are plotted on a computer-generated picture of ∂K_{-1}.

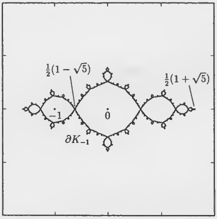


Figure 2.12 ∂K_{-1}

This picture suggests that the repelling fixed points $\frac{1}{2}(1 \pm \sqrt{5})$ lie on the boundary of K_{-1}, whereas the super-attracting 2-cycle $0, -1$ lies in the interior of K_{-1}. A similar phenomenon occurred for P_0 in the solution to Problem 2.7(b). The super-attracting fixed point 0 of P_0 lies in the interior of K_0, whereas the repelling fixed point at 1 and the 2-cycle and the 3-cycles, which are all repelling, lie on the boundary of K_0. In fact, the following general result holds.

See Problem 2.8(b), for example.

> **Theorem 2.5** Let α be a periodic point of the function P_c.
>
> (a) If α is attracting, then α is an interior point of K_c.
>
> (b) If α is repelling, then α is a boundary point of K_c.

The definitions of interior point and boundary point are given in *Unit A3*, Subsection 5.1.

Proof

This proof may be omitted on a first reading.

(a) Suppose that α is an attracting periodic point of P_c, with period p. Then α is an attracting fixed point of the pth iterate P_c^p. Hence, by Theorem 1.1, there is an open disc with centre α whose points are attracted to α under iteration of P_c^p. These points therefore do not escape to ∞ under iteration of P_c, and so they must lie in K_c. Hence α is an interior point of K_c.

(b) First suppose that α is a repelling fixed point, so that

$$P_c(\alpha) = \alpha \quad \text{and} \quad |P'_c(\alpha)| > 1.$$

Since $\alpha \in K_c$, we need to show that α is *not* an interior point of K_c. If it were an interior point, then we could choose an open disc $\{z : |z-\alpha| < r\}$ lying in K_c. In that case

Since K_c is closed, $\partial K_c = K_c - \operatorname{int} K_c$.

$$P_c^n(z) \in K_c, \qquad \text{for } |z-\alpha| = \tfrac{1}{2}r \text{ and } n = 1, 2, \ldots,$$

by Theorem 2.3(d), and hence

$$|P_c^n(z)| \le r_c, \qquad \text{for } |z-\alpha| = \tfrac{1}{2}r \text{ and } n = 1, 2, \ldots,$$

$r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$

by Theorem 2.3(a). Now we apply Cauchy's Estimate to each of the polynomial functions P_c^n to deduce that

Unit B2, Problem 3.3

$$\left|(P_c^n)'(\alpha)\right| \le \frac{r_c}{\frac{1}{2}r} = \frac{2r_c}{r}, \qquad \text{for } n = 1, 2, \ldots. \tag{2.11}$$

By the Chain Rule, as in the proof of Theorem 2.4,

$$\begin{aligned}(P_c^n)'(\alpha) &= P_c'(P_c^{n-1}(\alpha)) \times \cdots \times P_c'(P_c(\alpha)) \times P_c'(\alpha)\\ &= (P_c'(\alpha))^n,\end{aligned}$$

since, by assumption, α is a fixed point of P_c. Because $|P_c'(\alpha)| > 1$, this implies that the sequence

$$\left|(P_c^n)'(\alpha)\right| = |P_c'(\alpha)|^n, \qquad n = 1, 2, \ldots,$$

tends to ∞, contrary to Estimate (2.11). Thus $\alpha \in \partial K_c$.

If α is a repelling periodic point of P_c, with period p, then a similar argument applies (with P_c^p rather than P_c); we omit the details. ■

Remarks

1 Notice in part (a) that each point of the cycle

$$\alpha,\ P_c(\alpha),\ P_c^2(\alpha), \ldots,\ P_c^{p-1}(\alpha),$$

is an attracting periodic point of P_c, with period p, by Theorem 2.4(b), and so each point of this cycle is an interior point of K_c. The effect of P_c is to map any point z near α to a point $P_c(z)$ near $P_c(\alpha)$, and so on, round and round the cycle (see Figure 2.13, in which, for convenience, each disc has the same radius).

If the sequence $z, P_c^p(z), P_c^{2p}(z), \ldots$ tends to α, then the sequence $P_c(z), P_c^{p+1}(z), P_c^{2p+1}(z), \ldots$ tends to $P_c(\alpha)$, by the continuity of P_c at α.

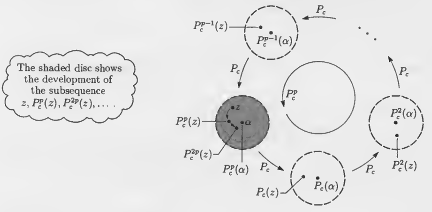


Figure 2.13

In this way the sequence $\{P_c^n(z)\}$ forms itself into p convergent subsequences, each converging to a point of the attracting p-cycle. In fact it can be shown that *every* interior point of K_c will be attracted in this way to the attracting p-cycle. See Figure 2.14, which shows an interior point z_0 of K_{-1} being attracted to the super-attracting 2-cycle $0, -1$.

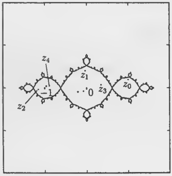


Figure 2.14
∂K_{-1} and $z_n = P_{-1}^n(z_0)$

2 In view of Theorem 2.5, it is natural to ask where in K_c do any *indifferent* periodic points of P_c lie. The answer is that it depends in a rather complicated way on the multiplier of the periodic point. This multiplier has modulus 1, and so it is of the form $e^{2\pi i a}$, where $0 \le a < 1$. It can be shown that if a is a

rational number, then the periodic point lies on ∂K_c, whereas if a is an irrational number, then the periodic point *usually* lies in the interior of K_c.

3 If we know that the set K_c has no interior points, for a given value of c, then Theorem 2.5(a) tells us that P_c can have no attracting periodic points. For example, $K_{-2} = \{x + iy : |x| \leq 2, y = 0\}$ has no interior points and so $P_{-2}(z) = z^2 - 2$ has no attracting periodic points, a fact which is not immediately obvious!

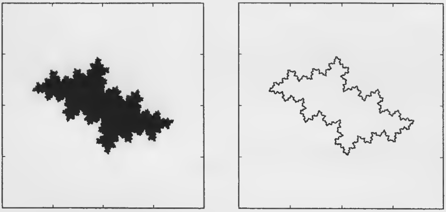
Problem 2.5

2.4 The Julia set of P_c

We saw in the previous subsection that the boundary of the keep set K_c (which is also the boundary of the escape set E_c) contains all the repelling periodic points of P_c. This set is of particular interest because it forms the 'watershed' between those points which escape to ∞ under iteration of P_c and those which are kept (see Figure 2.15). It is called the *Julia set* of P_c, in honour of the mathematician who first studied it in detail.

This section is intended for reading only.

Gaston Julia (1893–1978), a French mathematician, developed the basic theory of the iteration of analytic functions while recovering from wounds received in World War I, and won a prize for this work from the Parisian Academy of Sciences. He was a professor at the Sorbonne, and worked on many aspects of complex analysis.



(a) $K_c : c = -0.5 + 0.5i$ (b) $J_c : c = -0.5 + 0.5i$

Figure 2.15

> **Definition** The **Julia set** J_c of P_c is the boundary of K_c.

As noted earlier, the set K_c has no holes in it and so K_c can be thought of as J_c together with the 'inside' of J_c. Hence K_c is often called the *filled-in Julia set*. For example,

$$K_0 = \{z : |z| \leq 1\}, \quad \text{so } J_0 = \partial K_0 = \{z : |z| = 1\},$$

and

$$K_{-2} = \{x + iy : |x| \leq 2, y = 0\}, \quad \text{so } J_{-2} = \partial K_{-2} = K_{-2}.$$

Notice that if K_c has no interior points then $J_c = K_c$.

A number of general properties of J_c can be deduced from Theorem 2.3. For each $c \in \mathbb{C}$, the Julia set J_c is a non-empty compact subset of $\{z : |z| \leq r_c\}$, which is completely invariant under P_c and symmetric under rotation by π about 0. Of these properties, only the complete invariance is a little tricky to prove, and this holds because of the following facts:

1 K_c is completely invariant (Theorem 2.3(d));

2 $\operatorname{int} K_c$ is completely invariant (by the continuity of P_c and the Open Mapping Theorem);

Unit C2, Theorem 3.1

3 $J_c = \partial K_c = K_c - \operatorname{int} K_c$.

The definition of J_c could be used to plot J_c, but there are also other methods. We have already seen that J_c contains all the repelling periodic points of P_c, and so a knowledge of a number of these points gives us some information about the shape of J_c. In fact it can be shown that

Theorem 2.5(b)

> J_c is the smallest closed set which contains all the repelling periodic points of P_c. (2.12)

Thus the shape of J_c is entirely determined by these points. However, calculating a large number of the repelling periodic points would be a difficult task in practice.

A more satisfactory method of plotting J_c is to use the complete invariance of this set under P_c. This complete invariance tells us that if $\alpha_1 \in J_c$, then the solutions of the equation $P_c(z) = \alpha_1$, that is, $z^2 + c = \alpha_1$, also lie in J_c. This equation has the solutions $\pm\sqrt{\alpha_1 - c}$, which are two new points, α_2, α_3 say, of J_c (see Figure 2.16). Now, however, we can repeat this process with α_2, α_3 instead of α_1 to obtain four new points α_4, α_5, α_6, α_7 in J_c. This process, which is illustrated schematically in Figure 2.17, is known as **backward iteration**. It can be shown that

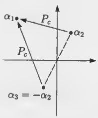


Figure 2.16

Exceptional cases occur when $\alpha_1 = c$, or when α_1 is a fixed point of P_c. In these cases, there is only one new point.

> J_c is the smallest closed set which contains all the backward iterates of any given point of J_c. (2.13)

Thus the shape of J_c is entirely determined by these backward iterates. The calculation of such backward iterates is not quite straightforward, even with a computer, because the tree-like structure shown in Figure 2.17 needs careful handling. A common short cut is to make a random choice of square root at each level and plot the resulting sequence; for example,

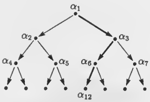
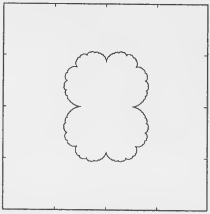


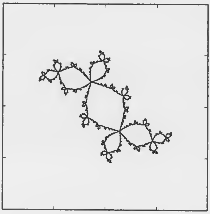
Figure 2.17

$$\alpha_1, \quad \alpha_3, \quad \alpha_6, \quad \alpha_{12}, \quad \ldots.$$

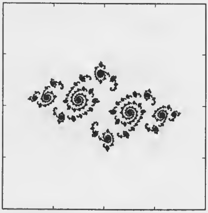
A convenient choice of starting point α_1 for the backward iteration is a repelling or indifferent fixed point of P_c. You can see the result of this method, for several values of c, in Figure 2.18.



(a) $J_c : c = 0.25$ (cauliflower)



(b) $J_c : c = -0.123 + 0.745i$ (rabbit)



(c) $J_c : c = -0.75 + 0.25i$ (sea-horse)

Figure 2.18

Remarks

1 Property (2.12) implies that near every point of J_c, we can find repelling periodic points. The effect of these repelling periodic points is to make the behaviour of the function P_c on J_c extremely unstable, in the sense that points near together on J_c tend to be pushed apart under the iteration of P_c. Such behaviour is often described as *chaotic*.

By contrast the behaviour of P_c on $\mathbb{C} - J_c$ is *stable*; that is, points close together in $\mathbb{C} - J_c$ behave in essentially the same way under iteration of P_c. This distinction between stable and unstable behaviour can be used to define the notion of a Julia set for *any* entire function or rational function.

2 Julia sets display a remarkable 'self-similarity' property. For each c, the shape of any part of J_c appears to be repeated all over J_c and is seen even when we zoom in closer and closer to J_c. This is a consequence of the complete invariance of J_c together with Property (2.13) of J_c. As a result, Julia sets are often described as *fractals.* The name fractal was introduced by B. Mandelbrot (of whom, more later) in 1975 to describe a type of set which is extremely irregular, and yet has an underlying structure that can be seen under successive magnifications of the set. The exact definition of a 'fractal' is still under discussion, but it has to do with certain methods of measuring the *dimension* of a set which may give non-integer answers!

3 GRAPHICAL ITERATION

After working through this section, you should be able to:

(a) use *graphical iteration* to determine the behaviour of real iteration sequences;

(b) describe properties of the keep sets K_c, for $c \in \mathbb{R}$, which can be obtained by using graphical iteration.

In this section we make some observations about the nature of the keep sets K_c when c is a *real* number. One simple observation is that if c is real, then K_c is symmetric under reflection in the real axis (see, for example, K_1 in Figure 2.4). This holds because, for $c \in \mathbb{R}$,

$$P_c(\overline{z}) = \overline{z}^2 + c = \overline{z^2 + c} = \overline{P_c(z)}, \qquad \text{for } z \in \mathbb{C},$$

so that, for $c \in \mathbb{R}$,

$$P_c^n(\overline{z}) \to \infty \text{ as } n \to \infty \quad \iff \quad P_c^n(z) \to \infty \text{ as } n \to \infty;$$

thus, $\overline{z} \in K_c$ if and only if $z \in K_c$.

We obtain some more interesting results by using a technique called graphical iteration, which applies only to the iteration of *real* functions.

3.1 What is graphical iteration?

If f is a real function, then an iteration sequence of the form $x_{n+1} = f(x_n)$ can be represented graphically by using the two graphs $y = f(x)$ and $y = x$ plotted together, as in Figure 3.1. Note that any point where $y = f(x)$ meets $y = x$ corresponds to a fixed point of f (for example, the point a in Figure 3.1).

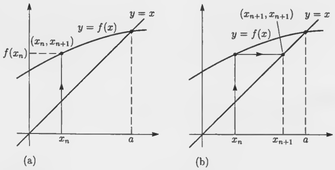


Figure 3.1

These two diagrams illustrate a two-stage process for finding geometrically the position on the x-axis of the term x_{n+1}, given the position of the term x_n:

(a) draw a vertical line to meet $y = f(x)$ at $(x_n, f(x_n)) = (x_n, x_{n+1})$;

(b) draw a horizontal line to meet $y = x$ at (x_{n+1}, x_{n+1}).

Given any initial term x_0, we can apply the above process repeatedly to construct the sequence $\{x_n\}$ geometrically, and thus obtain information about the behaviour of $\{x_n\}$. For example, with the function f in Figure 3.1 and with $x_0 = 0$, we obtain the behaviour illustrated in Figure 3.2, which strongly suggests that $\{x_n\}$ tends to the fixed point a.

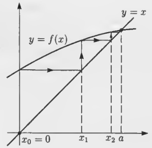


Figure 3.2

In the following example we carry out graphical iteration with a particular function f.

Example 3.1

Let $f(x) = \frac{1}{2}x + 1$.

(a) Plot $y = f(x)$ and $y = x$ on the same diagram and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$ and $x_0 = 3$.

(b) Describe the behaviour of each of these sequences $\{x_n\}$ and check that your answer agrees with the solution to Problem 1.9(b)(i).

Solution

(a) The graphs $y = f(x)$ and $y = x$ are plotted in Figure 3.3. They meet at the point $(2, 2)$, which corresponds to 2, the only fixed point of f. The sequences $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$, with $x_0 = 0$ and $x_0 = 3$, are also plotted.

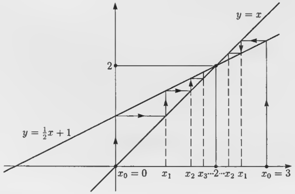


Figure 3.3

(b) For both values of x_0, the figure strongly suggests that $x_n \to 2$ as $n \to \infty$. In Problem 1.9(b)(i) we found that if $|a| < 1$, then the iteration sequence $z_{n+1} = az_n + b$, $n = 0, 1, 2, \ldots$, converges to (the fixed point) $\alpha = b/(1 - a)$ for all z_0. Here we have $a = \frac{1}{2}$, $b = 1$ and $\alpha = 2$, so our answer agrees with this result. ■

The next problem gives you a chance to try out graphical iteration.

Problem 3.1

Let $f(x) = -2x + 1$.

(a) Plot $y = f(x)$ and $y = x$ on the same diagram and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$ and $x_0 = \frac{1}{3}$.

(b) Describe the behaviour of each of these sequences $\{x_n\}$ and check that your answer agrees with the solution to Problem 1.9(b)(i).

3.2 Real quadratic iteration sequences

We now apply graphical iteration to real quadratic iteration sequences of the form

$$x_{n+1} = P_c(x_n) = x_n^2 + c, \qquad n = 0, 1, 2, \ldots,$$

where c is real. A great deal can be said about sequences of this special form, but we confine attention to a few basic results which throw some light on the corresponding keep sets K_c.

To illustrate the method, we consider the case $c = 0$. The graphs $y = P_0(x) = x^2$ and $y = x$ meet at $(0,0)$ and $(1,1)$, corresponding to the fixed points 0, 1 of P_0, and the iterations plotted in Figure 3.4 indicate that if $|x| \le 1$ then

$$0 \le P_0^n(x) \le 1, \qquad \text{for } n = 1, 2, \ldots.$$

On the other hand, if $|x| > 1$, then

$$P_0^n(x) \to \infty \text{ as } n \to \infty.$$

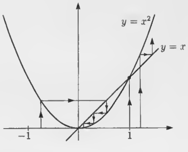


Figure 3.4

These results show that the part of the keep set $K_0 = \{z : |z| \le 1\}$ on the real axis is the closed interval $[-1, 1]$, as expected.

Now the only values of c for which we know K_c explicitly are $c = 0$ and $c = -2$. For other *real* values of c, we may expect that graphical iteration will give new information about K_c. To see what kind of information can be obtained in this way, try the following problem.

Problem 3.2

(a) Plot $y = x^2 + 1$ and $y = x$ on the same diagram and apply graphical iteration to the sequence $x_{n+1} = x_n^2 + 1$, $n = 0, 1, 2, \ldots$, with your own choice of initial term x_0.

(b) Explain why the sequence $\{x_n\}$ tends to infinity.

(c) What do you deduce about the set K_1?

The solution to Problem 3.2 suggests that the presence or absence of real fixed points of $P_c(x) = x^2 + c$ is of fundamental importance to the behaviour of iteration sequences of the form $x_{n+1} = x_n^2 + c$. Since the fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, the following lemma is evident.

Problem 2.3

> **Lemma 3.1** If $c \in \mathbb{R}$, then the function P_c has
> (a) no real fixed points if $c > \frac{1}{4}$;
> (b) the single fixed point $\frac{1}{2}$, if $c = \frac{1}{4}$;
> (c) the two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, if $c < \frac{1}{4}$.

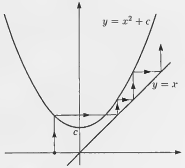


Figure 3.5 $c > \frac{1}{4}$

Part (a) of Lemma 3.1 shows that if $c > \frac{1}{4}$, then the graph $y = x^2 + c$ lies entirely above $y = x$ (see Figure 3.5). It follows by graphical iteration that if $c > \frac{1}{4}$, then

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \qquad \text{for all } x \in \mathbb{R}. \tag{3.1}$$

An algebraic proof of (3.1) is as follows.

If $x_{n+1} = x_n^2 + \frac{1}{4} + \varepsilon$, where $\varepsilon > 0$, then

$$x_{n+1} - x_n = \left(x_n - \tfrac{1}{2}\right)^2 + \varepsilon \ge \varepsilon;$$

hence

$$x_n \ge x_0 + n\varepsilon$$

and so $x_n \to \infty$ as $n \to \infty$.

Thus, if $c > \frac{1}{4}$, then all real values of x escape to ∞ under iteration of P_c, and so no real values of x belong to the keep set K_c. This gives the following result.

> **Theorem 3.1** If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \varnothing$.

It follows, by Theorem 2.3, parts (c) and (e), that if $c > \frac{1}{4}$, then K_c must have points in both the upper and lower open half-planes. Thus, for these values of c, the set K_c is in at least two pieces, and so is not connected. Moreover, the plot of K_1 in Figure 2.4 suggests that the set K_1 does not meet the imaginary axis. We ask you to prove this for $K_c, c > \frac{1}{4}$, in the next problem.

Problem 3.3

(a) Show that if c is real and y is real, then $P_c(iy)$ is real.

(b) Deduce from part (a) and Theorem 3.1 that if $c > \frac{1}{4}$, then K_c does not meet the imaginary axis.

If $c \le \frac{1}{4}$, then, by Lemma 3.1, P_c has either one or two real fixed points and so the keep set K_c does meet the real axis. It turns out that if c lies in the interval $\left[-2, \frac{1}{4}\right]$, then the set $K_c \cap \mathbb{R}$ is precisely equal to the symmetric closed interval

$$I_c = \left[-\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c},\ \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}\right].$$

Note that $I_0 = [-1, 1]$ and $I_{-2} = [-2, 2]$, as expected.

> **Theorem 3.2** If $-2 \le c \le \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$.

Proof The graphs $y = P_c(x)$ and $y = x$ are shown in Figure 3.6, together with a square S with sides parallel to the axes, which meets both axes in the interval I_c. The key to the proof is the observation that if $-2 \le c \le \frac{1}{4}$, then the points of the graph $y = P_c(x)$, for $x \in I_c$, lie in S. If $0 \le c \le \frac{1}{4}$, then this is evident because $y = P_c(x)$ does not extend below the x-axis. If $-2 \le c \le 0$, then we need to show that the lowest point of $y = P_c(x)$ does not lie below the bottom edge of the square S, that is,

$$c \ge -\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c}, \qquad \text{for } -2 \le c \le 0. \tag{3.2}$$

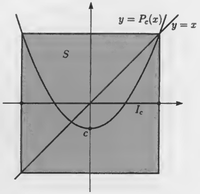


Figure 3.6

This inequality can be verified directly or we can use Figure 2.2. This shows that

$$|c| \le r_c = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}, \qquad \text{for } 0 \le |c| \le 2,$$

and hence that

$$-c \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c}, \qquad \text{for } -2 \le c \le 0,$$

which gives Inequality (3.2).

It follows that

$$\begin{aligned} x \in I_c &\Longrightarrow P_c^n(x) \in I_c, \quad \text{for } n = 1, 2, \ldots, \\ &\Longrightarrow x \in K_c. \end{aligned}$$

Moreover, graphical iteration shows that

$$\begin{aligned} x \notin I_c &\Longrightarrow P_c^n(x) \to \infty \text{ as } n \to \infty \\ &\Longrightarrow x \notin K_c. \end{aligned}$$

Thus $K_c \cap \mathbb{R} = I_c$. ■

If $c = -2$, then we know that

$$K_{-2} = \{x + iy : |x| \le 2, y = 0\} = I_{-2}.$$

However, it can be shown that for other values of c in the interval $\left[-2, \frac{1}{4}\right]$, the keep set K_c does not lie entirely on the real axis.

For $c < -2$, the keep set K_c has a more interesting intersection with the real axis, which is described in the next result.

Theorem 3.3 If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c from which a sequence of disjoint, non-empty, open subintervals of I_c has been removed. In particular, $0 \notin K_c$.

Proof Let S be the square used in the proof of Theorem 3.2, which meets the axes in the interval I_c. First note that, by graphical iteration,

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \qquad \text{for } x \in \mathbb{R} - I_c, \tag{3.3}$$

so that $K_c \cap \mathbb{R} \subseteq I_c$.

Now, since $c < -2$, the lowest point on $y = P_c(x)$ lies below S, as in Figure 3.7. Therefore, the set A_0 of points in I_c which escape from I_c after exactly one iteration of P_c,

$$A_0 = \{x \in I_c : P_c(x) \notin I_c\},$$

is an open subinterval of I_c with centre 0 (see Figure 3.7). In view of (3.3), it follows that

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \qquad \text{for } x \in A_0,$$

and so the points of A_0 do not lie in K_c. In particular, $0 \notin K_c$.

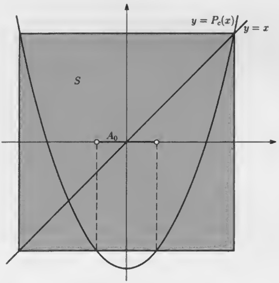


Figure 3.7

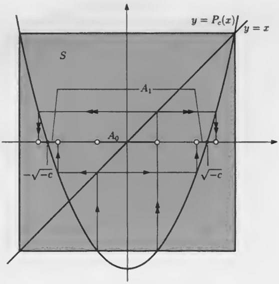


Figure 3.8

We now consider the set A_1 of points in I_c which remain in I_c for exactly one iteration of P_c:

$$\begin{aligned} A_1 &= \{x \in I_c : P_c(x) \in I_c, \text{ but } P_c^2(x) \notin I_c\} \\ &= \{x \in I_c : P_c(x) \in A_0\}. \end{aligned}$$

The set A_1 (see Figure 3.8) consists of two open subintervals of I_c which are positioned symmetrically on either side of 0 and contain the points $\pm\sqrt{-c}$, which are the zeros of P_c.

The construction lines with single and double arrowheads in Figure 3.8 indicate how the endpoints of the two open subintervals, which comprise A_1, are found.

More generally, we consider the set A_n of points in I_c which remain in I_c for exactly n iterations of P_c, defined inductively as follows:

$$A_n = \{x \in I_c : P_c(x) \in A_{n-1}\}, \qquad \text{for } n = 1, 2, \ldots .$$

Since each open interval in A_{n-1} gives rise to two open intervals in A_n, it follows that the set A_n consists of 2^n disjoint open subintervals of I_c. Now, any point of I_c which escapes to ∞ under iteration of P_c must lie in exactly one of the sets A_n. Thus the sets A_n are disjoint and

This can be *proved* by using Mathematical Induction.

$$K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \cdots),$$

which gives the required structure. ■

Remarks

1 The set $K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \cdots)$ is infinite because it contains all the endpoints of all the intervals which comprise A_n, for $n = 0, 1, 2 \ldots$. Actually, there are infinitely many points in $K_c \cap \mathbb{R}$ which are *not* endpoints of this type, but these are harder to identify.

2 It can be shown that if $c < -2$, then $K_c \subseteq \mathbb{R}$ (as is the case for $c = -2$).

Problem 3.4

Show that if $c < -2$, then K_c does not meet the imaginary axis.
(*Hint*: Problem 3.3(a) and Figure 3.7 should help.)

Problem 3.4 shows that if $c < -2$, then the set K_c is in at least two pieces, and so (as when $c > \frac{1}{4}$) it is not connected. We pursue the question of the connectedness of K_c in the next section.

4 THE MANDELBROT SET

After working through this section, you should be able to:

(a) understand the definition of the *Mandelbrot set* M;

(b) use the Fatou–Julia Theorem and its corollaries to determine whether certain points lie in M;

(c) appreciate how a computer may be used to plot M;

(d) show that certain points c lie in M because the corresponding function P_c has an *attracting cycle*;

(e) appreciate where certain periodic regions of the Mandelbrot set are located by making use of *saddle-node bifurcations* and *period-multiplying bifurcations*.

4.1 What is the Mandelbrot set?

In Section 2 we gave computer-generated plots of a number of keep sets K_c for quadratic functions of the form $P_c(z) = z^2 + c$. The shapes of these sets are remarkably varied, but they can be classified into two distinct types: those which are 'all in one piece' (for example, K_0 and K_{-2}) and those which are 'in more than one piece' (for example, K_1). The mathematical name for a set which is 'all in one piece' is *connected.*

Earlier in the course we introduced *pathwise connectedness*, in order to define the key concept of a *region.* It is clear that the sets K_0 and K_{-2} are both pathwise connected, but it is not so evident that the complicated set K_{-1} is pathwise connected, even though it does appear to be in one piece. We find it convenient to introduce the following more general notion of connectedness.

Unit A3, Section 4

Definitions A set A is **disconnected** if there are disjoint open sets G_1 and G_2 such that

$$A \cap G_1 \neq \varnothing, \quad A \cap G_2 \neq \varnothing \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$

A set A is **connected** if it is not disconnected.

Throughout the rest of this unit, this is the meaning of the word *connected.*

For example, for each real $c > \frac{1}{4}$, the set K_c is disconnected because, by Theorem 3.1 and Problem 2.3(a), it does not meet the real axis, but it does have points in both the upper and lower open half-planes. Thus, for $c > \frac{1}{4}$, the definition of disconnected is satisfied with $A = K_c$, $G_1 = \{z : \operatorname{Im} z > 0\}$, and $G_2 = \{z : \operatorname{Im} z < 0\}$; see Figure 4.1, where $c = \frac{1}{2}$.

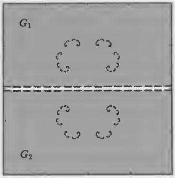


Figure 4.1 $K_{1/2}$

Problem 4.1

Use the result of Problem 3.4 to show that K_c is disconnected for real $c < -2$.

It is more difficult to prove that a connected set *is* connected. This is because we need to show that it is not disconnected, that is, the open sets G_1, G_2 in the above definition *do not exist.* However, it is not too difficult to show that

any pathwise connected set is connected

(although we omit the details), and so the sets K_0 and K_{-2} are connected. At present, we set aside the difficulty of proving that a set is connected and

However a connected set need not be pathwise connected.

concentrate on the set of points c such that K_c is connected; this set is the celebrated *Mandelbrot set.*

Benoit Mandelbrot was born in Warsaw in 1924. His initial training as a mathematician took place at the Ecole Polytechnique, Paris. He started working for IBM in 1958 and became an IBM Fellow in 1974, at the Watson Research Institute in New York State.

Definition The **Mandelbrot set** is the set M of complex numbers c such that K_c is connected.

Remark This definition is often phrased in terms of the connectedness of the Julia set J_c. By using the fact that K_c has no holes in it, it can be shown that K_c is connected if and only if J_c is connected.

For example, since K_0 and K_{-2} are connected, we have $0 \in M$ and $-2 \in M$. On the other hand, by Problem 4.1 and the discussion preceding it, we have $c \notin M$, for all real $c > \frac{1}{4}$ and $c < -2$.

The definition of M is typical of many in mathematics. It defines the object that we are interested in precisely, but as it stands it is difficult to work with. For most values of c we have very little idea what K_c looks like, let alone whether or not it is connected! Fortunately, however, there is a numerical method of deciding whether K_c is connected, based on the following fundamental result, the proof of which is outlined in Subsection 4.3.

Theorem 4.1 Fatou–Julia Theorem

For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

Pierre Fatou (1878–1929) was a French astronomer, although he trained as a mathematician. In addition to his work on iteration theory, he proved an important result about the boundary behaviour of complex functions. Fatou and Julia (see page 27) proved (independently) a more general version of this result in 1918–19. After their pioneering work in complex iteration, there were only a few other developments in this field until the explosion of interest in the 1980s, sparked off by the use of computers.

This result states that the keep set K_c is connected if and only if the point 0 does not escape to ∞ under iteration of P_c. For example, 0 lies in both K_0 and K_{-2}, which are connected, but 0 does not lie in K_1, which is disconnected.

To see the power of Theorem 4.1, try the following problem, which characterizes that part of the Mandelbrot set which lies on the real axis.

Problem 4.2

(a) Use Theorem 4.1 and Theorem 3.2 to show that if $c \in \left[-2, \frac{1}{4}\right]$, then $c \in M$.

(b) Deduce that $M \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$.

It is natural to ask why the number 0 appears in this special way in Theorem 4.1. The reason, as you will see in Subsection 4.3, is that the number 0 is the only **critical point** of each of the functions $P_c(z) = z^2 + c$; that is, it is the only point at which each of the derivatives $P_c'(z) = 2z$ vanishes. By the Local Mapping Theorem, an analytic function fails to be one-one near a critical point, and so such points play a significant role in the function's behaviour.

Unit C2, Theorem 3.2

By using Theorem 2.3, we can turn the condition $0 \in K_c$ in Theorem 4.1 into a numerical condition which is easier to check. By Theorem 2.3(d),

$$0 \in K_c \implies P_c^n(0) \in K_c, \text{ for } n = 0, 1, 2, \ldots,$$

and so, by Theorem 2.3(a),

$$0 \in K_c \implies |P_c^n(0)| \le r_c, \text{ for } n = 0, 1, 2, \ldots, \tag{4.1}$$

where $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$.

Since $P_c(0) = c$, the right-hand side of Implication (4.1) shows that $|c| \le r_c$ and hence, from Figure 2.2, that $r_c \le 2$. Thus

$$0 \in K_c \implies |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots . \tag{4.2}$$

Now, if c satisfies the inequalities on the right-hand side of Implication (4.2), then the sequence $\{P_c^n(0)\}$ is bounded and so $0 \in K_c$, by the definition of K_c. Hence, by the definition of M and Theorem 4.1,

$$\begin{aligned} c \in M &\iff K_c \text{ is connected} \\ &\iff 0 \in K_c \\ &\iff |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots, \end{aligned}$$

and so we obtain the following corollary to Theorem 4.1.

Corollary 1 The Mandelbrot set M can be specified as follows:

$$M = \{c : |P_c^n(0)| \le 2, \text{ for } n = 1, 2, \ldots\}.$$

Remark Note that if $c \notin M$, then $0 \notin K_c$, by Theorem 4.1, and so $P_c^n(0) \to \infty$ as $n \to \infty$.

Corollary 1 provides a numerical criterion for determining whether or not a point c belongs to M. For any given c, we simply compute the terms of the sequence $\{P_c^n(0)\}$, which are

$$c, c^2 + c, (c^2 + c)^2 + c, \ldots,$$

and try to decide whether *all* these terms lie in $\{z : |z| \le 2\}$.

For $n = 1, 2, \ldots,$
$$P_c^{n+1}(0) = (P_c^n(0))^2 + c.$$

For example, if $c = 0$, then $\{P_c^n(0)\}$ is the constant sequence

$$0, 0, 0, \ldots;$$

since all these terms lie in $\{z : |z| \le 2\}$, we deduce by Corollary 1 that $0 \in M$. On the other hand, if $c = 1$, then the terms of $\{P_c^n(0)\}$ are

$$1, 2, 5, 26, \ldots;$$

since these terms do not all lie in $\{z : |z| \le 2\}$, we deduce by Corollary 1 that $1 \notin M$.

Problem 4.3

Use Corollary 1 to determine which of the following points lie in M.

(a) $c = -2$ (b) $c = 1 + i$ (c) $c = i$ (d) $c = \sqrt{2}\, i$

Corollary 1 makes it possible to use a computer to plot an approximation to M. A naive algorithm involves checking the inequality

$$|P_c^n(0)| \le 2 \tag{4.3}$$

for a large number of points c, and for $n = 1, 2, \ldots, N$, where N is a suitably large positive integer. If Inequality (4.3) is false for some n, then the corresponding c lies outside M, but if it is true for $n = 1, 2, \ldots, N$, then c must be in M or 'close to M'. The set M was first plotted in 1979 using an algorithm of this kind by Mandelbrot (who had previously been plotting Julia sets).

A plot of the 'periodic regions' of M (see Subsection 4.2) was made in 1978 by R. Brooks and J. P. Matelski. They had encountered the iteration of quadratic functions while studying various groups of Möbius transformations.

A rendering of the set M using this approach is shown in Figure 4.2. As you can see, the set appears to be very complicated, consisting of many 'blobs' (the main one of which is bounded by a cardioid), which Mandelbrot called *atoms*, arranged in a highly organized manner. Some of these atoms are stuck together to form a complex *molecule*, whereas others appear to float free.

The box represented in Figure 4.2 is

$$\{c : -2 \le \operatorname{Re} c < 1,\ -1.5 \le \operatorname{Im} c \le 1.5\}.$$

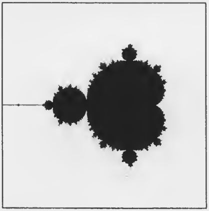


Figure 4.2 M — naive algorithm

Note that M is in the c-plane, often called the *parameter plane*, whereas each keep set K_c lies in the z-plane, often called the *dynamical plane.*

We can use Corollary 1 obtain a number of basic properties of M.

Corollary 2 The Mandelbrot set M

(a) is a compact subset of $\{c : |c| \le 2\}$;

(b) is symmetric under reflection in the real axis;

(c) meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$;

(d) has no holes in it.

Statement (d) means that $\mathbb{C} - M$ is connected.

Proof First note that each term of the sequence $\{P_c^n(0)\}$ defines a polynomial function of c. Indeed

This proof may be omitted on a first reading.

$$P_c(0) = c,$$
$$P_c^2(0) = c^2 + c,$$
$$P_c^3(0) = \left(c^2 + c\right)^2 + c = c^4 + 2c^3 + c^2 + c,$$

and, in general, $P_c^n(0)$ takes the form

$$P_c^n(0) = c^{2^{n-1}} + 2^{n-2}c^{2^{n-1}-1} + \cdots + c^2 + c, \qquad \text{for } n \ge 1. \tag{4.4}$$

This form of $P_c^n(0)$ can be justified by Mathematical Induction.

(a) To prove part (a), we define the sets

$$M_n = \{c : |P_c^n(0)| \le 2\}, \qquad \text{for } n = 1, 2, \ldots,$$

so that $M_1 = \{c : |c| \le 2\}$, $M_2 = \left\{c : \left|c^2 + c\right| \le 2\right\}$, and so on. Then, by Corollary 1,

$$M = M_1 \cap M_2 \cap \cdots,$$

and, in particular, $M \subseteq M_1 = \{c : |c| \le 2\}$. Thus M is bounded.

Each of the sets M_n is closed, because its complement

$$\mathbb{C} - M_n = \{c : |P_c^n(0)| > 2\}$$

is open. Indeed, if $\left|P_{c_0}^n(0)\right| > 2$ for some c_0, then this inequality must hold for all c in some open disc with centre c_0, by the continuity of the function $c \longmapsto |P_c^n(0)|$. It follows that M itself must be closed, because if $c \notin M$, then $c \notin M_n$ for some n, and so some open disc with centre c must lie outside M_n and hence outside M. Hence M is closed and bounded; that is, M is compact.

See the proof of Theorem 2.3(b) for similar reasoning.

(b) Because $P_c^n(0)$ is a polynomial in c with real coefficients,

$$P_{\bar{c}}^n(0) = \overline{P_c^n(0)} \implies |P_{\bar{c}}^n(0)| = |P_c^n(0)|, \text{ for } n = 1, 2, \ldots.$$

Hence, by Corollary 1, $\bar{c} \in M$ if and only if $c \in M$, and so M is symmetric under reflection in the real axis.

(c) You proved this in Problem 4.2(b).

(d) The proof that $\mathbb{C} - M$ is connected is very similar to the proof of part (f) of Theorem 2.3, using $\{c : |c| > 2\}$ instead of $\{z : |z| > r_c\}$ and applying the Maximum Principle to the analytic function $c \longmapsto P_c^n(0)$ instead of to $z \longmapsto P_c^n(z)$. We omit the details. ■

It turns out that the picture of the Mandelbrot set in Figure 4.2 is misleading. Some parts of M are so thin that the naive algorithm fails to detect them. This became clear when A. Douady and J. H. Hubbard proved the following remarkable result in 1982.

It was Douady and Hubbard who named the set M after Mandelbrot.

Theorem 4.2 The Mandelbrot set is connected.

We make no attempt to prove Theorem 4.2.

Thus it follows that all parts of M in Figure 4.2 are actually linked together. Knowing this fact, we can devise more effective algorithms for plotting better renderings of M, such as the one shown (with various enlargements corresponding to the small square boxes) in Figure 4.3.

We discuss the structure of M in Subsection 4.2.

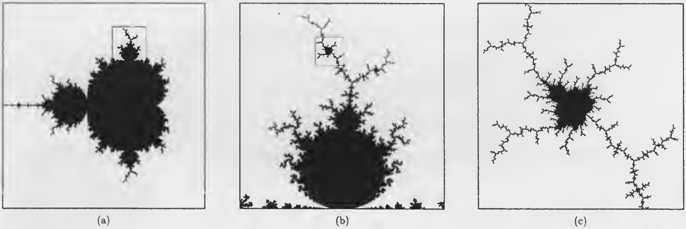


Figure 4.3 M — showing its connectedness

To gain some insight into why the connectedness of M is so remarkable, we note that the sets $M_n = \{c : |P_c^n(0)| \leq 2\}, n = 1, 2, \ldots$, defined in the proof of Corollary 2, are in fact nested, that is,

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots.$$

We can represent these nested sets M_n readily using a computer. See Figure 4.4, where, for the first few values of n, points of $M_n - M_{n+1}$ are plotted black if n is odd and white if n is even. Since $c \in M_n - M_{n+1}$ if and only if

$$\left|P_c^k(0)\right| \leq 2, \qquad \text{for } k = 1, 2, \ldots, n, \text{ but } \left|P_c^{n+1}(0)\right| > 2,$$

these bands measure how long the corresponding sequences $\{P_c^n(0)\}$ remain in $\{z : |z| \leq 2\}$. It appears that each of the boundaries

$$\partial M_n = \{c : |P_c^n(0)| = 2\}, \qquad n = 1, 2, \ldots,$$

forms a simple-closed smooth path (that is, it does not break up into several pieces). This surprising fact can be shown to be equivalent to the connectedness of M.

Figure 4.4

By Equation (4.4),

$$\partial M_1 = \{c : |c| = 2\},$$
$$\partial M_2 = \{c : \left|c^2 + c\right| = 2\},$$

and so on.

4.2 Inside the Mandelbrot set

Corollary 1 of Theorem 4.1 provides a good means of showing that a given point c is *not* in the set M. This corollary is not so helpful, however, as a means of checking that c is in M. Of course, for some values of c, such as $c = 0$ or $c = -2$, it is possible to check directly that $|P_c^n(0)| \leq 2$, for $n = 1, 2, \ldots$, but this is usually not the case. Instead, the following general result can often be used.

> **Theorem 4.3** If the function P_c has an attracting cycle, then $c \in M$.

Remark One way to prove Theorem 4.3 is to show that if P_c has an attracting cycle, then the sequence $\{P_c^n(0)\}$ is attracted to this cycle in the way described in Remark 1 following Theorem 2.5. Since $\{P_c^n(0)\}$ can be attracted in this way to at most one cycle, it follows that P_c has at most one attracting cycle for each value of c.

We outline another proof of Theorem 4.3 in Subsection 4.3.

Theorem 4.3 allows us to identify various key parts of M, by determining for which values of c the function P_c has an attracting p-cycle for various values of p. The solution for attracting fixed points and attracting 2-cycles is particularly elegant, as we now show.

> **Theorem 4.4**
>
> (a) The function P_c has an attracting fixed point if and only if c satisfies
>
> $$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3. \qquad (4.5)$$
>
> (b) The function P_c has an attracting 2-cycle if and only if c satisfies
>
> $$|c+1| < \tfrac{1}{4}. \qquad (4.6)$$

As you will see in the proof of Theorem 4.4(a), the Condition (4.5) is equivalent to the statement that c lies inside the cardioid with parametrization

$$\gamma(t) = \tfrac{1}{2}e^{it} - \tfrac{1}{4}e^{2it} \qquad (t \in [-\pi, \pi]).$$

This cardioid is the boundary of the main 'atom' of M. Condition (4.6) means that c lies inside the open disc $\{c : |c+1| < \frac{1}{4}\}$ which lies immediately to the left of the cardioid; see Figure 4.5. Thus if a point c lies in one of these two sets, then it is possible to verify this using Theorem 4.4, and hence to show that $c \in M$. For example, the point $c = -\frac{1}{2} + \frac{1}{2}i$, shown in Figure 4.5, seems to lie just inside the cardioid. For this value of c, $|c|^2 = \frac{1}{2}$ and $\mathrm{Re}\,c = -\frac{1}{2}$, and so

$$\begin{aligned}\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c &= \left(4 - \tfrac{3}{2}\right)^2 - 8 \cdot \tfrac{1}{2} \\ &= \tfrac{9}{4} < 3.\end{aligned}$$

Hence P_c has an attracting fixed point, by Theorem 4.4(a), and so $c = -\frac{1}{2} + \frac{1}{2}i$ lies in M, by Theorem 4.3.

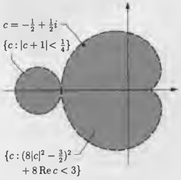


Figure 4.5

An alternative approach is to show that $|P_c'(\alpha)| < 1$ for one of the fixed points α of P_c, but it is messier.

Problem 4.4

Prove that each of the following points lies in M.

(a) $c = -0.9 + 0.1i$ (b) $c = 0.2 + 0.5i$

Proof of Theorem 4.4(a)

We ask you to prove part (b) in Problem 4.5.

First note that α is a fixed point of P_c if and only if

$$P_c(\alpha) = \alpha^2 + c = \alpha,$$

that is, if and only if

$$c = \alpha - \alpha^2.$$

Moreover, this fixed point is attracting if and only if

$$|P_c'(\alpha)| = |2\alpha| < 1.$$

Thus P_c has an attracting fixed point if and only if c is of the form $\alpha - \alpha^2$, where $|\alpha| < \frac{1}{2}$, that is, if and only if c lies in the image of the open disc $\{z : |z| < \frac{1}{2}\}$ under the function $z \longmapsto z - z^2$. To understand the nature of this image we use the approach of *Unit D1*, Section 4, expressing the function $z \longmapsto z - z^2$ as a composition of one-one conformal mappings. In fact,

$$\begin{aligned} z - z^2 &= \tfrac{1}{4} - \left(\tfrac{1}{4} - z + z^2\right) \\ &= \tfrac{1}{4} - \left(\tfrac{1}{2} - z\right)^2, \end{aligned}$$

and so the function $z \longmapsto z - z^2$ has the effect indicated in Figure 4.6.

Note that the cusp of the cardioid arises because the function $z \longmapsto z - z^2$ is two-one near the point $\frac{1}{2}$ (see *Unit C2*, Problem 3.3). In particular, this function doubles angles between smooth paths emerging from $\frac{1}{2}$.

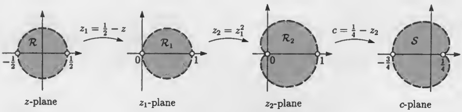


Figure 4.6

To complete the proof of part (a), we need to show that c lies in the region $\mathcal{S}$, bounded by the cardioid if and only if

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c < 3.$$

This is a rather fiddly technical exercise, which can be done in various ways. One approach is to note that the cardioid itself is the image of the circle $|z| = \frac{1}{2}$ under the function $z \longmapsto z - z^2$, that is, it is the path in the c-plane with parametrization

$$\begin{aligned} \gamma(t) &= \tfrac{1}{2}e^{it} - \left(\tfrac{1}{2}e^{it}\right)^2 \\ &= \tfrac{1}{2}(\cos t + i\sin t) - \tfrac{1}{4}(\cos 2t + i\sin 2t) \qquad (t \in [-\pi, \pi]). \end{aligned}$$

In *Unit A2*, Exercise 2.6, we found that this path has equation

$$4|c|^4 - \tfrac{3}{2}|c|^2 + \tfrac{1}{2}\,\mathrm{Re}\,c = \tfrac{3}{64},$$

which can be rearranged to give

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c = 3.$$

It can then be shown that, for $\frac{1}{4} < r < \frac{3}{4}$, the cardioid splits the part of the circle $|c| = r$ in the open upper half-plane into two arcs (see Figure 4.7). On the left arc, which is inside the cardioid, Inequality (4.5) holds, whereas on the right arc we have the opposite inequality. For other positive values of r, the cardioid does not meet this semi-circle, and the direction of the inequality can be found by considering its nature at $c = \pm r$. We omit the details. ■

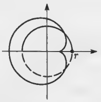


Figure 4.7

Problem 4.5

(a) Prove that

$$P_c^2(z) - z = (P_c(z) - z)(z^2 + z + c + 1).$$

(b) Deduce from part (a) that if $c \neq -\frac{3}{4}$, then P_c has the 2-cycle α_1, α_2 where

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{-\tfrac{3}{4} - c}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{-\tfrac{3}{4} - c},$$

with multiplier

$$(P_c^2)'(\alpha_1) = 4\alpha_1\alpha_2.$$

What happens if $c = -\frac{3}{4}$?

(c) Deduce from part (b) that P_c has an attracting 2-cycle if and only if $|c + 1| < \frac{1}{4}$.

This proves Theorem 4.4(b).

Theorem 4.4 suggests that each atom in the Mandelbrot set has an associated period p such that, for each c in the interior of the atom, the function P_c has an attracting p-cycle. Numerical experiments appear to confirm this and so we introduce the notion of a *periodic region*.

Definition A **periodic region** is a maximal region $\mathcal{R}$ such that, for some positive integer p,

$$\text{the function } P_c \text{ has an attracting } p\text{-cycle, for all } c \in \mathcal{R}. \qquad (*)$$

The word 'maximal' signifies that there is no region satisfying $(*)$ which contains $\mathcal{R}$ but which is not equal to $\mathcal{R}$.

For example, the inside of the cardioid and the open disc specified by Inequalities (4.5) and (4.6) are both periodic regions. Unfortunately, none of the other periodic regions in M seems to have such a straightforward characterization, but we can obtain some information about their location by using the following result.

Theorem 4.5 The function P_c has a super-attracting p-cycle if and only if

$$P_c^p(0) = 0, \quad \text{but } P_c^k(0) \neq 0, \quad \text{for } k = 1, 2, \ldots, p-1. \qquad (4.7)$$

Proof By Theorem 2.4, any p-cycle of P_c

$$\alpha, P_c(\alpha), \ldots, P_c^{p-1}(\alpha),$$

has multiplier

$$P_c'(\alpha)P_c'(P_c(\alpha)) \cdots P_c'(P_c^{p-1}(\alpha)) = (2\alpha)(2P_c(\alpha)) \cdots (2P_c^{p-1}(\alpha)),$$

since $P_c'(z) = 2z$. Therefore, such a p-cycle is super-attracting if and only if one of the points of the p-cycle is 0. But P_c has a p-cycle including 0 if and only if Condition (4.7) holds, and so the proof is complete. ■

For example, if $p = 1$, then Condition (4.7) becomes

$$P_c(0) = c = 0,$$

as expected, since $P_0(z) = z^2$ has the super-attracting fixed point 0.

If $p = 2$, then Condition (4.7) becomes

$$P_c^2(0) = c^2 + c = 0, \quad \text{but } P_c(0) = c \neq 0.$$

The only solution is $c = -1$, as expected, since $P_{-1}(z) = z^2 - 1$ has the super-attracting 2-cycle $0, -1$.

Problem 4.6

Show that the function P_c has a super-attracting 3-cycle for precisely three different points c, one of which lies in the interval $[-1.8, -1.7]$ and the other two of which form a pair of complex conjugates.

It can be proved that if P_c has an attracting p-cycle, then c lies in a periodic region which contains exactly one point c_0, say, with a super-attracting p-cycle. We call c_0 the **centre** of the associated periodic region. Figure 4.8 shows the approximate location in the Mandelbrot set of all points c_0 for which P_{c_0} has a super-attracting p-cycle, for $p = 1, 2, 3, 4$.

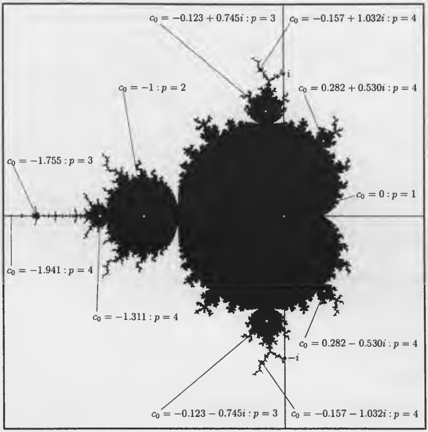


Figure 4.8

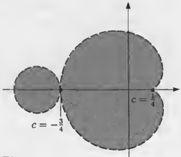


Figure 4.9

In fact the boundaries of all the periodic regions are either (roughly) circular in shape or are shaped like a cardioid but they are linked together in a very complicated manner. We cannot hope to give a full explanation of the way in which the periodic regions of M fit together, but we can gain some insight into their structure by looking more closely at the two sets given by Theorem 4.4. These are plotted in Figure 4.9.

The key points to consider here are $c = \frac{1}{4}$, which is the cusp of the cardioid, and $c = -\frac{3}{4}$, where the cardioid and the circle meet. These are the points where the cycle structure of P_c changes; for example, as c passes through $-\frac{3}{4}$ from the cardioid into the disc, the attracting fixed point of P_c becomes repelling and the repelling 2-cycle of P_c becomes attracting. At such points, we say that a *bifurcation* occurs. In order to characterize such bifurcations, we look more closely at the cycle structure of P_c for $c = \frac{1}{4}$ and $c = -\frac{3}{4}$.

For $c = \frac{1}{4}$, the function $P_c = P_{1/4}$ has just one fixed point, $\alpha = \frac{1}{2}$, with multiplier

$$P_c'(\alpha) = P_{1/4}'\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{2}\right) = 1.$$

For $c = -\frac{3}{4}$, the function $P_c = P_{-3/4}$ has two fixed points, one of which is at $\alpha = -\frac{1}{2}$, with multiplier

$$P'_c(\alpha) = P'_{-3/4}(-\tfrac{1}{2}) = 2(-\tfrac{1}{2}) = -1.$$

Thus, for both these key points c, the function P_c has a fixed point α whose multiplier $P'_c(\alpha)$ is a root of unity. In order to state a general result, we say that the number λ is a **primitive nth root of unity** if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$. For example, -1 is a primitive square root of unity, but it is not a *primitive* fourth root of unity.

-1 is the multiplier we found above for $c = -\frac{3}{4}$ and $\alpha = -\frac{1}{2}$.

The following description gives the two types of bifurcation which occur when the multiplier of a cycle is a root of unity.

Theorem 4.6 Suppose that the function $P_{c_0}, c_0 \in \mathbb{C}$, has a p-cycle whose multiplier λ is a root of unity.

(a) **Saddle-node bifurcation at c_0** If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region $\mathcal{R}$, such that

$$P_c \text{ has an attracting } p\text{-cycle, for } c \in \mathcal{R}.$$

(b) **Period-multiplying bifurcation at c_0** If λ is a primitive nth root of unity, for $n > 1$, then there are two periodic regions $\mathcal{R}_1$ and $\mathcal{R}_2$ whose boundaries meet at c_0 such that

$$P_c \text{ has an attracting } \begin{cases} p\text{-cycle}, & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle}, & \text{for } c \in \mathcal{R}_2. \end{cases}$$

We omit the proof of this theorem.

The name *saddle-node* bifurcation arises from the shape of the graph when such a bifurcation occurs within a family of *real* functions.

If $n = 2$, the name 'period-doubling bifurcation' is also used.

In the next two problems, which are quite challenging, we ask you to investigate specific examples of these two types of bifurcations.

Problem 4.7

(a) Use your solution to Problem 2.2(a) to determine a polynomial $Q_c(z)$ such that

$$P_c^3(z) - z = (P_c(z) - z)Q_c(z).$$

(b) Verify that

$$Q_{-7/4}(z) = \left(z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8}\right)^2.$$

(c) Deduce that a saddle-node bifurcation occurs at $c = -\frac{7}{4}$, and relate this fact to the solution of Problem 4.6 and to Figure 4.8.

Problem 4.8

Show that if $c = \zeta - \zeta^2$, where 2ζ is a root of unity ($\neq 1$), then a period-multiplying bifurcation occurs at c. Relate this fact to Figure 4.8, with $\zeta = -\frac{1}{2}, \frac{1}{2}e^{2\pi i/3}$ and $\frac{1}{2}i$.

(*Hint:* First check that ζ is a fixed point of P_c.)

We are now in a better position to describe the structure of the Mandelbrot set M. Using the approach of Problem 4.8, we find that each part of the main cardioid in Figure 4.8 is decorated by periodic regions. In a similar way all these periodic regions are themselves decorated everywhere by further periodic regions, and so on.

The rest of this subsection is intended for reading only.

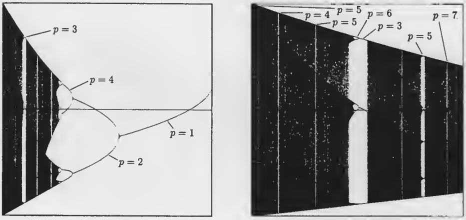
All these period-multiplying bifurcations go a long way towards explaining the complicated structure of M. In addition, however, we find throughout the boundary of M the appearance of small cardioid-shaped periodic regions, arising from saddle-node bifurcations, such as the one at $c = -\frac{7}{4}$ in Problem 4.7. All these cardioid-shaped regions are themselves decorated with smaller periodic regions, as a result of period-multiplying bifurcations, and so

they give rise to small copies of the Mandelbrot set within itself (see Figure 4.3(c)). These copies appear to be linked together in M (remember that the Mandelbrot set is connected) by a complicated network of 'veins'. The simplest such vein lies along the real axis from -2 to $\frac{1}{4}$. In fact, one of the unsolved problems about the Mandelbrot set M is to decide whether M is not only connected, but also *pathwise* connected.

A pathwise connected set must be connected, but a connected set need not be pathwise connected. However, for open sets the two types of connectedness are equivalent.

The real bifurcation diagram

Finally, we describe a very effective graphical method of obtaining information about the periodic regions of the Mandelbrot set whose centres lie on the real axis. The idea is to choose a large number of real values of c between -2 and $\frac{1}{4}$, and plot each of the corresponding sequences $\{P_c^n(0)\}$ vertically above and below a horizontal c-axis (see Figure 4.10(a)). In order to determine any attracting p-cycles (to which this sequence will be attracted, by the remark following Theorem 4.3) the first 200 or so terms are discarded and the next 600 or so are plotted. Thus if P_c has an attracting p-cycle, then p points should be plotted above and below the corresponding point c.



(a) $-2 \leq c \leq \frac{1}{4}$ (b) $-2 \leq c \leq -1.54$

Figure 4.10

As expected, this 'bifurcation diagram' reveals the convergence of $\{P_c^n(0)\}$ to an attracting fixed point for $-\frac{3}{4} < c < \frac{1}{4}$, to an attracting 2-cycle for $-\frac{5}{4} < c < -\frac{3}{4}$, to an attracting 4-cycle for c just to the left of $-\frac{5}{4}$, with further period-doubling bifurcations to the left of this. Also visible is an attracting 3-cycle for c just to the left of $-\frac{7}{4}$. For other values of c, it is less clear what is happening, but by scaling the c-axis appropriately (see Figure 4.10(b)) many other 'periodic windows' are revealed which correspond to attracting p-cycles, some of which are labelled in the diagram. Notice that in each such window there is a point c whose cycle includes 0; this value of c is the centre of the corresponding periodic region in the Mandelbrot set. In fact, pictures of this type for the related family of iteration sequences given by

$$x_{n+1} = k\,x_n(1 - x_n), \text{ with } x_0 = \tfrac{1}{2}, \quad \text{where } 0 \leq k \leq 4,$$

were studied in the early 1970s, before the Mandelbrot set itself had been plotted. In particular, the order in which the periodic windows appear was found and the rate at which periodic-doubling occurs was discovered (by M. Feigenbaum) to have a certain universal property. Thus, even the part of the Mandelbrot set which lies on the real axis is extremely complicated, and there are still unanswered questions about it. For example, it is believed that every non-empty open interval of $[-2, 1/4]$ meets at least one of the periodic windows, but at the time of writing (1993) this has still not been proved.

4.3 Outline proofs of Theorems 4.1 and 4.3

The aim of this subsection is to indicate, without going into all the details, why Theorems 4.1 and 4.3 are true.

This subsection may be omitted on a first reading.

Theorem 4.1 Fatou–Julia Theorem

For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

Theorem 4.3 If the function P_c has an attracting cycle, then $c \in M$.

We shall need the concept of the **preimage set** $P_c^{-1}(E)$ of a set E under the function P_c. This is simply the set of points which are mapped to E by P_c:

$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

This notation does *not* imply that P_c has an inverse function.

For example, if $E = \{-1\}$ and $c = 0$, then $P_0^{-1}(\{-1\}) = \{i, -i\}$. Note that $P_c^{-1}(E)$ is always symmetric under rotation by π about 0, since P_c is an even function.

We say that a **compact disc** is a compact set whose boundary is a simple-closed smooth path (see Figure 4.11). Note that a compact disc need not be a disc, but it must be connected. The nature of the preimage set of a compact disc E under P_c is given by the following result (see Figure 4.12).

Figure 4.11

Lemma 4.1 If E is a compact disc and $c \notin \partial E$, then $P_c^{-1}(E)$ is

(a) one compact disc containing 0, if $c \in \operatorname{int} E$;

(b) two compact discs, neither containing 0, if $c \in \operatorname{ext} E$.

The interior and exterior of a subset of $\mathbb{C}$ are defined in *Unit A3*, Section 5.

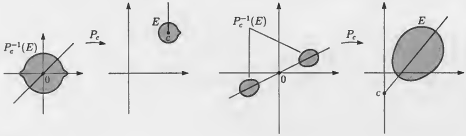


Figure 4.12

If $c \in \partial E$, then $P_c^{-1}(E)$ forms a filled-in figure of eight.

To indicate why Lemma 4.1 is true we have shown in both parts of Figure 4.12 a typical ray emerging from the point c and meeting the set E, as well as the preimage of this ray which consists of the two rays emerging from 0, combining to form a line through 0.

Now we start the outline proof of Theorem 4.1 by choosing a closed disc $E_0 = \{z : |z| \le r\}$, where

$$r > \max\{r_c, |c|\} \tag{4.8}$$

$r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$

and

$$P_c^n(0) \notin \partial E_0, \text{ for } n = 1, 2, \ldots. \tag{4.9}$$

Then we define the sequence of successive preimage sets of E_0 under P_c:

$$E_1 = P_c^{-1}(E_0), \quad E_2 = P_c^{-1}(E_1), \quad \text{and so on,}$$

Note that the sets $E_0, E_1, E_2, \ldots$ are not escape sets here.

that is, we put

$$E_n = \{z : P_c^n(z) \in E_0\}, \qquad \text{for } n = 1, 2, \dots .$$

If $P_c^{n+1}(z) \in E_0$, then we must have $P_c^n(z) \in E_0$ also (by Theorem 2.2, because $r > r_c$). Hence

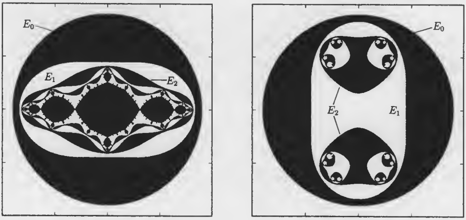
$$E_{n+1} \subseteq E_n, \qquad \text{for } n = 0, 1, 2, \dots,$$

so that the sets E_n are nested. Moreover,

$$\begin{aligned} E_0 \cap E_1 \cap E_2 \cap \cdots &= \{z : P_c^n(z) \in E_0, \text{ for } n = 0, 1, 2, \dots\} \\ &= \{z : |P_c^n(z)| \le r, \text{ for } n = 0, 1, 2, \dots\} \\ &= K_c, \end{aligned} \tag{4.10}$$

by Theorem 2.2. Thus the shape of K_c is determined by the shapes of the sets E_n.

Figure 4.13(a) and (b) illustrate the first few of these nested sets E_n, $n = 1, 2, \dots$, for the two cases c $= -1$ and c $= 1$. In both cases we have taken $E_0 = \{z : |z| \le r\}$, where $r = 1.8$, so that Conditions (4.8) and (4.9) are satisfied in each case. Points of $E_n - E_{n+1}$ are plotted black if n is even and white if n is odd.



(a) $K_c : c = -1$ (b) $K_c : c = 1$

Figure 4.13

In Figure 4.13(a), *all* the sets E_n are compact discs. However, in Figure 4.13(b) the sets E_0 and E_1 are compact discs, but E_2 consists of two compact discs, E_3 consists of four compact discs, and so on. We now show that these different structures are related to whether or not the point 0 (and hence c) lies in K_c.

First assume that $0 \in K_c$, so that c $= P_c(0) \in K_c$. Then

$$c \in \operatorname{int} E_n, \qquad \text{for } n = 0, 1, 2, \dots,$$

so that, by Lemma 4.1(a),

$$E_{n+1} = P_c^{-1}(E_n) \text{ is one compact disc}, \qquad \text{for } n = 0, 1, 2, \dots .$$

Note that

$$c \notin \partial E_n, \text{ for } n = 0, 1, 2, \dots,$$

since $P_c^n(c) = P_c^{n+1}(0) \notin \partial E_0$, for $n = 0, 1, 2, \dots$, by Condition (4.9).

Thus E_n, $n = 0, 1, 2, \dots$, is a nested sequence of connected compact sets (as in Figure 4.13(a)), and it follows from this (though we do not give the details) that $K_c = E_0 \cap E_1 \cap \cdots$ is also connected, as required.

Next assume that $0 \notin K_c$, so that c $= P_c(0) \notin K_c$. Since c $\in \operatorname{int} E_0$, there is a positive integer m such that

$$c \in \operatorname{int} E_{m-1}, \quad \text{but } c \notin E_m.$$

Thus, by Lemma 4.1(b), E_m is one compact disc, but E_{m+1} is two; see Figure 4.14 (which is based on Figure 4.13(b)), in which c $= 1$ and $m = 1$.

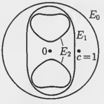


Figure 4.14

Thus, the set K_c lies in the union of the two halves of E_{m+1}, and it has points in each half because both K_c and E_{m+1} are symmetric under rotation by π about 0. Hence K_c is disconnected, as required. This completes the outline proof of Theorem 4.1.

Notice from Figure 4.14 that c lies outside both halves of E_{m+1} and so, by Lemma 4.1(b), the set $E_{m+2} = P_c^{-1}(E_{m+1})$ consists of four compact discs (two for each half of E_{m+1}). In general, E_{m+n} consists of 2^n compact discs. Now, K_c is a subset of each of these preimage sets E_{m+n}, and so any connected subset $\widetilde{K}_c$ of K_c must lie in exactly one of the 2^n compact discs comprising E_{m+n}, say $\widetilde{E}_{m+n}$. Thus

$$\widetilde{E}_{m+1} \supseteq \widetilde{E}_{m+2} \supseteq \cdots \supseteq \widetilde{K}_c.$$

It can be proved that the diameters of these compact discs $\widetilde{E}_{m+n}$ tend to zero as $n \to \infty$, so that $\widetilde{K}_c$ must be a singleton set, and there will be one such singleton set for each of the (infinitely many) possible nested sequences of compact discs $\widetilde{E}_{m+n}$. A set obtained by a construction of this type is called a **Cantor set**, and so K_c is such a Cantor set for each c not in M.

This requires a rather tricky application of the Riemann Mapping Theorem and Schwarz's Lemma.

Georg Cantor (1845–1918) developed the foundations of 'set theory'. For example, he showed that the set of rational numbers is countable (that is, its elements can be arranged to form a sequence), but the set of real numbers is uncountable. Any Cantor set is uncountable.

It follows that if $c \notin M$, then K_c has no interior points and so the function P_c cannot have an attracting cycle, in view of Theorem 2.5(a). This shows one possible way to prove Theorem 4.3.

5 BEYOND THE MANDELBROT SET

After reading through this section, you should be able to:

(a) appreciate the universal nature of the Mandelbrot set.

This section is intended for reading only.

In Subsection 1.4 we discussed briefly the Newton–Raphson function

$$N(z) = \frac{2z^3 + 1}{3z^2}, \tag{5.1}$$

corresponding to the cubic polynomial function $p(z) = z^3 - 1$.

Under iteration of N, all points of $\mathbb{C}$ are attracted to one of the zeros of p, or else they remain on the common basin boundary (see Figure 1.9), which includes the point at ∞ for the extended function $\widehat{N}$.

Following the discovery of the Mandelbrot set, the Newton–Raphson method for a general cubic function was investigated by computer, in the early 1980s. For most cubic functions, the corresponding Newton–Raphson function behaves under iteration in the same way as the function in Equation (5.1), but for some cubic functions a difference was found.

To make this difference precise, we consider the family of cubic functions given by

$$\begin{aligned} p_c(z) &= (z-1)\left(z + \tfrac{1}{2} - c\right)\left(z + \tfrac{1}{2} + c\right) \\ &= z^3 - \left(\tfrac{3}{4} + c^2\right) z - \tfrac{1}{4} + c^2, \end{aligned} \tag{5.2}$$

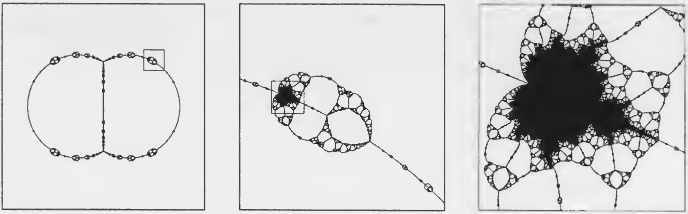
For example, if $c = \pm(\sqrt{3}/2)i$, then

$$p_c(z) = z^3 - 1.$$

where $c \in \mathbb{C}$. The corresponding Newton–Raphson function is

$$N_c(z) = z - \frac{p_c(z)}{p_c'(z)} = \frac{2z^3 + \left(\frac{1}{4} - c^2\right)}{3z^2 - \left(\frac{3}{4} + c^2\right)},$$

and it is a straightforward matter to check that the critical points of N_c (that is, the points where N_c' vanishes) are the three zeros of p_c *and* the point 0. Usually, the critical point 0 is attracted to one of the zeros of p_c under iteration of N_c, although it may also remain on the basin boundary (for example, if $c = \pm(\sqrt{3}/2)i$, then $\widehat{N}_c(0) = \infty$ and $\widehat{N}_c(\infty) = \infty$). For some values of c, however, the function N_c has an attracting p-cycle, where $p > 1$, to which the point 0 is attracted. In Figure 5.1(a), we have plotted in the parameter plane those values of c for which the sequence $\{N_c^n(0)\}$ does *not* converge to one of the zeros of p_c.



(a) (b) (c)

Figure 5.1 $\{c : N_c^n(0) \nrightarrow \text{a zero of } p_c\}$

If we zoom in on various parts of this set, then we find small copies of the Mandelbrot set! Inspired by examples of this type, Douady and Hubbard developed a theory which shows that copies of the Mandelbrot set appear in the parameter plane whenever we consider the iteration of suitable families of analytic functions. Thus the Mandelbrot set has a universal nature!

The boxes represented in Figures 5.1(a) and 5.2(a) are each

$$\{c : -2 \le \operatorname{Re} c, \operatorname{Im} c \le 2\}.$$

To emphasize this universal nature, we describe one relation of the Mandelbrot set, which is obtained by iterating *non-analytic* functions of the form

$$f_c(z) = \bar{z}^2 + c,$$

where $c \in \mathbb{C}$. By analogy with Theorem 4.1, Corollary 1, we plot the set

$$\{c : |f_c^n(0)| \le 2, \text{ for } n = 0, 1, 2, \ldots\},$$

which is called the **tricorn**, or **Mandelbar set**; see Figure 5.2(a).

(a) (b) (c)

Figure 5.2 The tricorn

This set is symmetric under rotation by $2\pi/3$ about 0 (as well as under reflection in the real axis), and it appears to be connected. Closer inspection reveals that some parts of the boundary of the tricorn are smooth (for example near $c = \frac{1}{4}$), whereas other parts are extremely irregular (see Figure 5.2(b)). As you might expect, the tricorn contains small copies of itself, but it also contains small copies of the Mandelbrot set — a truly universal object!

In this unit we have only scratched the surface of the subject of complex iteration. For example, there is much more to be said about the structure of the individual Julia sets J_c, and there are many families of entire functions (such as $z \longmapsto e^{cz}$, where c is a complex parameter), whose behaviour under iteration leads to completely new phenomena. Nevertheless, we hope that you have gained some insight into this remarkable subject, and that you appreciate the irony in the following quotation from A. Douady.

> I must say that, in 1980, whenever I told my friends that I was just starting with J. H. Hubbard a study of polynomials of degree 2 in one complex variable (and more specifically those of the form $z \longmapsto z^2 + c$), they would all stare at me and ask: Do you expect to find anything new?

From H.-O. Pietgen and P. H. Richter, *The Beauty of Fractals* (Springer-Verlag, 1986).

EXERCISES

Section 1

Exercise 1.1 Plot the terms z_0, z_1, z_2, z_3 for each of the following iteration sequences, and write down the corresponding function f.

(a) $z_{n+1} = iz_n, \qquad z_0 = 2i$

(b) $z_{n+1} = 2z_n(1 - z_n), \qquad z_0 = \frac{1}{4}$

(c) $z_{n+1} = \dfrac{z_n}{z_n + 1}, \qquad z_0 = \frac{1}{2}(-1 + i)$

Exercise 1.2 Find all the fixed points of each of the following functions and classify them as (super-)attracting, repelling or indifferent.

(a) $f(z) = z - z^2$ (b) $f(z) = 2z(1 - z)$

(c) $f(z) = z^2 - \frac{1}{2}$ (d) $f(z) = z/(z + 1)$

Exercise 1.3 Prove that any iteration sequence of the form

$$z_{n+1} = 2z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots,$$

is conjugate, via the conjugating function $h(z) = 1 - 2z$, to one of the form

$$w_{n+1} = w_n^2, \qquad n = 0, 1, 2, \ldots.$$

Hence obtain a formula for z_n in terms of z_0.

Exercise 1.4 Let $f(z) = (az + b)/(cz + d)$, where $ad - bc \neq 0$ and $c \neq 0$, let $\widehat{f}$ be the corresponding extended Möbius transformation, and let $z_n = \widehat{f}^n(z_0)$, where $z_0 \in \widehat{\mathbb{C}}$.

This exercise, which is fairly challenging, contains a complete analysis of the behaviour of iteration sequences defined by Möbius transformations.

(a) Prove that if $(a - d)^2 + 4bc = 0$, then $\widehat{f}$ has a unique fixed point α, which lies in $\mathbb{C}$. Deduce that the sequence $\{z_n\}$ is conjugate, via the conjugating function $h(z) = 1/(z - \alpha)$, to an iteration sequence of the form

$$w_{n+1} = w_n + 2c/(a + d), \qquad n = 0, 1, 2, \ldots.$$

Hence prove that $z_n \to \alpha$ as $n \to \infty$, for all $z_0 \in \widehat{\mathbb{C}}$.

(b) Prove that if $(a - d)^2 + 4bc \neq 0$, then $\widehat{f}$ has two fixed points α and β, both lying in $\mathbb{C}$. Deduce that $\{z_n\}$ is conjugate, via the conjugating function $h(z) = (z - \alpha)/(z - \beta)$, to an iteration sequence of the form

$$w_{n+1} = f'(\alpha)w_n, \qquad n = 0, 1, 2, \ldots.$$

Hence prove that if $|f'(\alpha)| < 1$, then $z_n \to \alpha$ as $n \to \infty$, for all $z_0 \in \widehat{\mathbb{C}} - \{\beta\}$. Also describe the behaviour of $\{z_n\}$ if

(i) $|f'(\alpha)| > 1$, (ii) $|f'(\alpha)| = 1$.

(*Hint*: You will need to use the fact that α and β are fixed points of f to simplify the algebra.)

Section 2

Exercise 2.1 Use Theorem 2.1 to show that the iteration sequence

$$z_{n+1} = 3z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots,$$

where $z_0 = \frac{1}{2}$, is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2 + d, \qquad n = 0, 1, 2, \ldots.$$

Exercise 2.2

(a) Prove that if $|c| \le \frac{1}{4}$, then

$$|z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|} \implies |P_c(z)| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}.$$

(b) Deduce from part (a) that if $|c| \le \frac{1}{4}$, then

$$\left\{ z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|} \right\} \subseteq K_c.$$

(c) Combine the result of part (b) with Theorem 2.3(a) to show that, when c is close to 0, the set K_c is approximately equal to the closed unit disc.

Exercise 2.3 For each of the following functions f and points α, show that α is a periodic point of f and decide whether it is (super-)attracting, repelling or indifferent.

(a) $f(z) = -z, \quad \alpha = i$ (b) $f(z) = z^2 - 2, \quad \alpha = \frac{1}{2}(-1 + \sqrt{5})$

(c) $f(z) = z^3 + i, \quad \alpha = 0$ (d) $f(z) = z^3, \quad \alpha = e^{\pi i/13}$

Exercise 2.4 This exercise relates to Properties (2.12) and (2.13) for the function $P_0(z) = z^2$, whose Julia set is the unit circle.

In order to attempt this exercise you will need to have read Subsection 2.4.

(a) Determine the repelling periodic points of P_0 and prove that each arc of the unit circle contains such a repelling periodic point.

(b) Determine the backward iterates under P_0 of the point 1 and prove that each arc of the unit circle contains such a backward iterate.

Section 3

Exercise 3.1 Let $P_{1/4}(x) = x^2 + \frac{1}{4}$.
Plot $y = P_{1/4}(x)$ and $y = x$ on the same diagram and use graphical iteration to plot the iteration sequence

$$x_{n+1} = P_{1/4}(x_n), \qquad n = 0, 1, 2, \ldots,$$

with $x_0 = 0$. Describe the behaviour of the sequence $\{x_n\}$.

Exercise 3.2

(a) Use graphical iteration to show that any iteration sequence of the form

$$x_{n+1} = \frac{x_n}{x_n + 1}, \qquad n = 0, 1, 2, \ldots,$$

with $x_0 \in \mathbb{R} - A$, where $A = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$, converges to the point 0. (Note that if $x = -1/n$, then $x/(x+1) = -1/(n-1)$.)

(b) Relate your answer to part (a) to Exercise 1.4(a).

Section 4

Exercise 4.1 Use Corollary 1 to Theorem 4.1 to determine which of the following points c lie in M.

(a) $c = -1 + 2i$ (b) $c = -1$ (c) $c = -1 + i$

Exercise 4.2 Show that if $|c| = 2$ but $c \neq -2$, then

$$\left|c^2 + c\right| > 2,$$

and deduce that $c \notin M$.
(*Hint*: Use the factorization $c^2 + c = c(c+1)$.)

Exercise 4.3 Let $M_n = \{c : |P_c^n(0)| \leq 2\}$, for $n = 1, 2, \ldots$.

(a) Prove that $M_n \subseteq M_1$, for $n = 2, 3, \ldots$.
(*Hint*: If $|c| > 2$, then $|c| > r_c$, by Figure 2.2.)

(b) Prove that $M_{n+1} \subseteq M_n$, for $n = 1, 2, \ldots$.
(*Hint*: If $|c| \leq 2$, then $2 \geq r_c$, by Figure 2.2.)

Exercise 4.4 Prove that each of the following points lies in M.

(a) $c = -1.1 - 0.1i$ (b) $c = 0.6i$

Exercise 4.5

(a) Prove that if $\mathcal{R}$ is a periodic region of M, then $\partial\mathcal{R} \subseteq M$.

(b) Deduce from part (a) that each of the following points lies in M.

(i) $c = \frac{1}{4} + \frac{1}{2}i$ (ii) $c = -1 + \frac{1}{4}i$

(*Hint*: Locate these points in Figure 4.8)

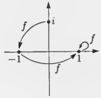
Exercise 4.6 Explain why there can be at most six values of c (see Figure 4.8) for which P_c has a super-attracting 4-cycle.

Exercise 4.7 Show that a period-doubling bifurcation occurs at $c = -\frac{5}{4}$, and relate this fact to Figure 4.8.

SOLUTIONS TO THE PROBLEMS

Section 1

1.1 (a) $i, -1, 1, 1$.



Here $f(z) = z^2$.

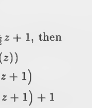
(b) $0, 1, \frac{3}{2}, \frac{7}{4}$.

Here $f(z) = \frac{1}{2}z + 1$.

(c) $0, -1, 0, -1$.

Here $f(z) = z^2 - 1$.

(d) $0, i, -1+i, -i$.



Here $f(z) = z^2 + i$.

1.2 If $f(z) = \frac{1}{2}z + 1$, then

$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f\left(\tfrac{1}{2}z + 1\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}z + 1\right) + 1 \\ &= \tfrac{1}{4}z + \tfrac{3}{2}. \end{aligned}$$

Also

$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= \tfrac{1}{4}\left(\tfrac{1}{2}z + 1\right) + \tfrac{3}{2} \\ &= \tfrac{1}{8}z + \tfrac{7}{4}. \end{aligned}$$

1.3 (a) If $f(z) = z + b$, then

$$f^1(z) = z + b,$$
$$f^2(z) = (z + b) + b = z + 2b,$$
$$f^3(z) = (z + 2b) + b = z + 3b,$$

and, in general,

$$f^n(z) = z + nb, \quad \text{for } n = 1, 2, \ldots.$$

(b) If $f(z) = z^3$, then

$$f^1(z) = z^3,$$
$$f^2(z) = \left(z^3\right)^3 = z^9,$$
$$f^3(z) = \left(z^9\right)^3 = z^{27},$$

and, in general,

$$f^n(z) = z^{3^n}, \quad \text{for } n = 1, 2, \ldots.$$

1.4 (a) If $z_0 = 1$, then we have

$$z_n = 1, \quad \text{for } n = 1, 2, \ldots,$$

so that $\{z_n\}$ is constant, and converges to 1.

(b) If $z_0 = -i$, then the terms of the sequence are

$$-i, -1, 1, 1, 1, \ldots,$$

so that $\{z_n\}$ is eventually constant, and converges to 1.

(c) If $z_0 = e^{2\pi i/3}$, then the terms of the sequence are

$$e^{2\pi i/3}, e^{4\pi i/3}, e^{8\pi i/3} = e^{2\pi i/3}, \ldots,$$

so that the sequence cycles endlessly between these two values.

(d) Since $z_n = z_0^{2^n}$ and $|1/z_0| < 1$, we deduce that

$$\frac{1}{z_n} = \left(\frac{1}{z_0}\right)^{2^n}, \quad n = 1, 2, \ldots,$$

is a null sequence. Hence $z_n \to \infty$ as $n \to \infty$, by the Reciprocal Rule (*Unit A3*, Theorem 1.5).

1.5 (a) The fixed point equation is $f(z) = \frac{1}{2}z + 1 = z$, and

$$\tfrac{1}{2}z + 1 = z \iff \tfrac{1}{2}z = 1,$$

so the only fixed point is 2.

(b) The fixed point equation is $f(z) = z^2 - 2 = z$, and

$$z^2 - 2 = z \iff z^2 - z - 2 = 0,$$

so the only fixed points are 2 and -1.

(c) The fixed point equation is $f(z) = z^3 = z$, and

$$z^3 = z \iff z\left(z^2 - 1\right) = 0,$$

so the only fixed points are 0, 1 and -1.

1.6 (a) Since $f'(z) = 2z$, we have

$|f'(0)| = 0 < 1$, so 0 is an attracting (in fact, a super-attracting) fixed point of f;

$|f'(1)| = 2 > 1$, so 1 is a repelling fixed point of f.

(b) Since $f'(z) = \frac{1}{2}$, we have

$|f'(2)| = \frac{1}{2} < 1$, so 2 is an attracting fixed point of f.

(c) Since $f'(z) = 2z$, we have

$|f'(2)| = 4 > 1$, so 2 is a repelling fixed point of f.

1.7 (a) From Example 1.2(a), we know that, for $n = 1, 2, \ldots,$

$$f^n(z) = \frac{1}{2^n}z$$
$$\to 0 \text{ as } n \to \infty, \text{ for all } z \in \mathbb{C}.$$

Hence the basin of attraction of 0 under f is

$$\{z : f^n(z) \to 0 \text{ as } n \to \infty\} = \mathbb{C}.$$

(b) From Problem 1.3(b), we know that, for $n = 1, 2, \ldots,$

$$f^n(z) = z^{3^n}.$$

Hence

$$f^n(z_0) \to 0 \text{ as } n \to \infty, \text{ for } |z_0| < 1,$$

but

$$f^n(z_0) \not\to 0 \text{ as } n \to \infty, \text{ for } |z_0| \geq 1.$$

Thus the basin of attraction of 0 under f is

$$\{z : f^n(z) \to 0 \text{ as } n \to \infty\} = \{z : |z| < 1\}.$$

1.8 First we note that the function $h(z) = -z + \frac{1}{2}$ is one-one on $\mathbb{C}$. If $w_n = h(z_n) = -z_n + \frac{1}{2}$, then $z_n = -w_n + \frac{1}{2}$, so

$$z_{n+1} = z_n - z_n^2, \quad n = 0, 1, 2, \ldots,$$

becomes

$$-w_{n+1} + \tfrac{1}{2} = \left(-w_n + \tfrac{1}{2}\right) - \left(-w_n + \tfrac{1}{2}\right)^2$$
$$= -w_n + \tfrac{1}{2} - w_n^2 + w_n - \tfrac{1}{4}, \quad \text{for } n = 0, 1, 2, \ldots,$$

that is,

$$w_{n+1} = w_n^2 + \tfrac{1}{4}, \quad \text{for } n = 0, 1, 2, \ldots,$$

which demonstrates that these two sequences are conjugate.

If $z_0 = \frac{1}{2}$, then $w_0 = 0$.

1.9 (a) First we note that the function

$$h(z) = z + b/(a-1),$$

where $a \neq 1$, is one-one on $\mathbb{C}$.

If $w_n = h(z_n) = z_n + b/(a-1)$, then $z_n = w_n - b/(a-1)$, so

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \ldots,$$

becomes

$$w_{n+1} - b/(a-1) = a(w_n - b/(a-1)) + b, \quad \text{for } n = 0, 1, 2, \ldots;$$

that is,

$$w_{n+1} = aw_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{1}$$

since $-b/(a-1) = -ab/(a-1) + b$.

(b) The iteration sequence (1) has general term $w_n = a^n w_0$ (see Example 1.2(a)). Hence

$$z_n = w_n - \frac{b}{a-1}$$
$$= a^n w_0 - \frac{b}{a-1}, \quad \text{for } n = 0, 1, 2, \ldots,$$

giving

$$z_n = a^n\left(z_0 + \frac{b}{a-1}\right) - \frac{b}{a-1}, \quad \text{for } n = 0, 1, 2, \ldots;$$

that is,

$$z_n = a^n z_0 - \frac{b(1-a^n)}{a-1}, \quad \text{for } n = 0, 1, 2, \ldots.$$

(i) If $|a| < 1$, then $\{a^n\}$ is a null sequence, so in this case

$$z_n \to \frac{b}{1-a} \text{ as } n \to \infty.$$

(Note that $b/(1-a)$ is the only fixed point of the function $f(z) = az + b$.)

(ii) If $|a| = 1$, $a \neq 1$, then $\{a^n\}$ is divergent, by *Unit A3*, Theorem 1.7(b). It follows that $\{z_n\}$ is divergent in this case (unless $z_0 = b/(1-a)$, in which case $\{a_n\}$ is constant).

(iii) If $|a| > 1$, then $\{a^n\}$ tends to infinity, by *Unit A3*, Theorem 1.7(a). It follows that $\{z_n\}$ tends to infinity in this case (unless $z_0 = b/(1-a)$).

1.10 With $N(z) = \dfrac{z^2 - b}{2z + a}$ and $h(z) = \dfrac{z - \alpha}{z - \beta}$, we have

$$h(N(z)) = \left(\frac{z^2 - b}{2z + a} - \alpha\right) \Big/ \left(\frac{z^2 - b}{2z + a} - \beta\right)$$
$$= \frac{z^2 - b - \alpha(2z + a)}{z^2 - b - \beta(2z + a)}$$
$$= \frac{z^2 - 2\alpha z - (a\alpha + b)}{z^2 - 2\beta z - (a\beta + b)}$$
$$= \frac{z^2 - 2\alpha z + \alpha^2}{z^2 - 2\beta z + \beta^2} \quad \text{(by the hint)}$$
$$= \frac{(z-\alpha)^2}{(z-\beta)^2}$$
$$= (h(z))^2,$$

as required.

1.11 If $p(z) = (z - \alpha)^2$, then

$$N(z) = z - \frac{(z-\alpha)^2}{2(z-\alpha)} = \tfrac{1}{2}(z + \alpha),$$

and so, by Problem 1.9(b)(i) (with $a = \frac{1}{2}, b = \frac{1}{2}\alpha$), all points of $\mathbb{C}$ are attracted to α under iteration of N.

Section 2

2.1 (a) By Theorem 2.1, with $a = -4$, $b = 4$, $c = 0$, the sequence $z_{n+1} = -4z_n^2 + 4z_n$ is conjugate to the sequence

$$w_{n+1} = w_n^2 + d,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = 2 - 4 = -2$, using $h(z) = -4z + 2$. In this case, $w_0 = h\left(\frac{1}{2}\right) = 0$.

(b) By Theorem 2.1, with $a = -2$, $b = 0$, $c = 1$, the sequence $z_{n+1} = 1 - 2z_n^2$ is conjugate to the sequence

$$w_{n+1} = w_n^2 + d,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = -2$, using $h(z) = -2z$. In this case, $w_0 = h(0) = 0$.

2.2 (a)

$$P_c^2(z) = P_c(P_c(z))$$
$$= \left(z^2 + c\right)^2 + c$$
$$= z^4 + 2cz^2 + c^2 + c;$$
$$P_c^3(z) = \left(z^4 + 2cz^2 + c^2 + c\right)^2 + c$$
$$= z^8 + 4cz^6 + \left(6c^2 + 2c\right)z^4 + \left(4c^3 + 4c^2\right)z^2 + c^4 + 2c^3 + c^2 + c.$$

(b) We have

$$\begin{aligned} P_c^{n+1}(z) &= P_c(P_c^n(z)) \\ &= (P_c^n(z))^2 + \mathrm{c}. \qquad (1) \end{aligned}$$

Since $P_c(z) = z^2 + \mathrm{c}$, it follows from Equation (1) that the degree of P_c^{n+1} is twice the degree of P_c^n, and hence the degree of P_c^n is 2^n. The evenness of P_c^n follows from the fact that P_c is even, because

$$P_c(-z) = (-z)^2 + \mathrm{c} = z^2 + \mathrm{c} = P_c(z).$$

2.3 **(a)** The fixed point equation is $P_c(z) = z^2 + \mathrm{c} = z$, and

$$z^2 + \mathrm{c} = z \iff z^2 - z + \mathrm{c} = 0.$$

Thus the fixed points are

$$\frac{1 \pm \sqrt{1 - 4\mathrm{c}}}{2} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \mathrm{c}}.$$

Let these fixed points be α and β, so that $\alpha + \beta = 1$, and

$$P_c'(\alpha) = 2\alpha \text{ and } P_c'(\beta) = 2\beta.$$

Now $\alpha + \beta = 1$ and so $\frac{1}{2}(P_c'(\alpha) + P_c'(\beta)) = 1$. Unless $\mathrm{c} = \frac{1}{4}$ (so that $P_c'(\alpha) = P_c'(\beta) = 1$), this implies that at least one of $P_c'(\alpha)$, $P_c'(\beta)$ lies outside the unit circle, and so the corresponding fixed point is repelling.

(In fact, if $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - \mathrm{c}}$, then $P_c'(\alpha)$ lies outside the unit circle.)

(b) If $\mathrm{c} = \frac{1}{4}$, then $P_c = P_{1/4}$ has just one fixed point, $\frac{1}{2}$, which is indifferent since $P_{1/4}'(\frac{1}{2}) = 1$.

2.4 **(a)**

$$r_0 = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + 0} = \tfrac{1}{2} + \tfrac{1}{2} = 1;$$

$$r_i = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |i|} = \tfrac{1}{2} + \sqrt{\tfrac{5}{4}} = \tfrac{1}{2}(1 + \sqrt{5});$$

$$r_{-2} = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |-2|} = \tfrac{1}{2} + \sqrt{\tfrac{9}{4}} = 2.$$

(b) The sequence $\{P_0^n(1)\}$ has terms $1, 1, 1, \ldots$, and so $P_0^n(1) \not\to \infty$ as $n \to \infty$. This shows that $r_0 = 1$ is the smallest value of r_0 for which Theorem 2.2 holds with $\mathrm{c} = 0$.

The sequence $\{P_{-2}^n(2)\}$ has terms $2, 2, 2, \ldots$, and so $P_{-2}^n(2) \not\to \infty$ as $n \to \infty$. This shows that $r_{-2} = 2$ is the smallest value of r_{-2} for which Theorem 2.2 holds with $\mathrm{c} = -2$.

2.5 **(a)** If $z_0 \in L$, then z_0 is real with $|z_0| \le 2$, and so

$$z_1 = z_0^2 - 2$$

is real with $-2 \le z_1 \le 2$, because $0 \le z_0^2 \le 4$; hence $z_1 \in L$. On repeating this process, we deduce that $z_n \in L$, for $n = 1, 2, \ldots$.

If $z_0 \in \mathbb{C} - L$, then $z_1 \in \mathbb{C} - L$. For if $z_1 \in L$, then

$$z_0^2 = z_1 + 2 \in [0, 4] \implies z_0 \in L.$$

On repeating this process, we deduce that

$$z_n \in \mathbb{C} - L, \quad \text{for } n = 1, 2, \ldots.$$

(b) Let $w_n = J^{-1}(z_n)$, for $n = 0, 1, 2, \ldots$, so that

$$|w_n| > 1, \quad \text{for } n = 0, 1, 2, \ldots.$$

Then $z_n = J(w_n)$, and so, for $n = 0, 1, 2, \ldots$, the equation $z_{n+1} = z_n^2 - 2$ gives

$$\begin{aligned} J(w_{n+1}) &= (J(w_n))^2 - 2 \\ &= \left(w_n + \frac{1}{w_n}\right)^2 - 2 \\ &= w_n^2 + \frac{1}{w_n^2} = J\left(w_n^2\right). \end{aligned}$$

Now the function J is one-one on $\{w : |w| > 1\}$ and hence

$$w_{n+1} = w_n^2, \quad \text{for } n = 0, 1, 2, \ldots.$$

Since $|w_0| > 1$, we have $w_n \to \infty$ as $n \to \infty$, and hence $z_n = w_n + 1/w_n \to \infty$ as $n \to \infty$ (by the Reciprocal Rule).

(c) By part (a), we find that no point of L belongs to E_{-2}, whereas by part (b) all points of $\mathbb{C} - L$ belong to E_{-2}. Hence $E_{-2} = \mathbb{C} - L$, and $K_{-2} = \mathbb{C} - E_{-2} = L$.

2.6 To prove that E_c is completely invariant under P_c, we note that

$$\begin{aligned} z \in E_c &\iff P_c^n(z) \to \infty \text{ as } n \to \infty \\ &\iff P_c^{n+1}(z) \to \infty \text{ as } n \to \infty \\ &\iff P_c^n(P_c(z)) \to \infty \text{ as } n \to \infty \\ &\iff P_c(z) \in E_c, \end{aligned}$$

as required.

2.7 **(a)** Since

$$P_i(-i) = (-i)^2 + i = -1 + i$$

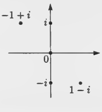
and

$$P_i(-1 + i) = (-1 + i)^2 + i = -i,$$

we have

$$P_i^2(-i) = -i \quad \text{and} \quad P_i^2(-1 + i) = -1 + i,$$

and so $-i$ and $-1 + i$ form a 2-cycle of P_i. Hence both these points lie in K_i as do i and $1 - i$, by Theorem 2.3(e). Another point lying in K_i is 0, because $P_i(0) = i$ and $i \in K_i$.



Clearly none of these points are fixed points of K_i.

(b) Since $P_0^3(z) = z^8$, we have to solve the equation $z^8 = z$:

$$\begin{aligned} z^8 = z &\iff z^8 - z = 0 \\ &\iff z\left(z^7 - 1\right) = 0. \end{aligned}$$

The solutions are 0 and $e^{2\pi k i/7}$, $k = 0, 1, \ldots, 6$ (*Unit A1*, Theorem 3.1).

Of these, the points 0 and 1 ($k = 0$) are fixed points of P_0, whereas

$$\begin{aligned} P_0\left(e^{2\pi i/7}\right) &= e^{4\pi i/7}, \\ P_0\left(e^{4\pi i/7}\right) &= e^{8\pi i/7}, \\ P_0\left(e^{8\pi i/7}\right) &= e^{16\pi i/7} = e^{2\pi i/7}, \end{aligned}$$

and

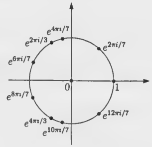
$$\begin{aligned} P_0\left(e^{6\pi i/7}\right) &= e^{12\pi i/7}, \\ P_0\left(e^{12\pi i/7}\right) &= e^{24\pi i/7} = e^{10\pi i/7}, \\ P_0\left(e^{10\pi i/7}\right) &= e^{20\pi i/7} = e^{6\pi i/7}. \end{aligned}$$

Hence

$$e^{2\pi i/7},\ e^{4\pi i/7},\ e^{8\pi i/7} \quad \text{and} \quad e^{6\pi i/7},\ e^{12\pi i/7},\ e^{10\pi i/7}$$

are both 3-cycles of P_0.

Using this and the results of Example 2.1, we obtain the following diagram.



(c) Since

$$\begin{aligned} P_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= \left(\tfrac{1}{2}(-1+\sqrt{2})\right)^2 - \tfrac{5}{4} \\ &= \tfrac{1}{4}(3-2\sqrt{2}) - \tfrac{5}{4} \\ &= \tfrac{1}{2}(-1-\sqrt{2}) \\ &\neq \tfrac{1}{2}(-1+\sqrt{2}), \end{aligned}$$

and

$$\begin{aligned} P^2_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= P_{-5/4}\left(\tfrac{1}{2}(-1-\sqrt{2})\right) \\ &= \left(\tfrac{1}{2}(-1-\sqrt{2})\right)^2 - \tfrac{5}{4} \\ &= \tfrac{1}{4}(3+2\sqrt{2}) - \tfrac{5}{4} \\ &= \tfrac{1}{2}(-1+\sqrt{2}), \end{aligned}$$

the point $\frac{1}{2}(-1+\sqrt{2})$ is a periodic point, with period 2, of $P_{-5/4}$. (It forms a 2-cycle with $\frac{1}{2}(-1-\sqrt{2})$.)

2.8 **(a)** The numbers $-i$, $-1+i$ form a 2-cycle of P_i, with multiplier

$$\begin{aligned} \left(P_i^2\right)'(-i) &= P_i'(-i)P_i'(-1+i) \\ &= (-2i)(-2+2i) \\ &= 4+4i, \end{aligned}$$

by Theorem 2.4. Hence

$$\left|\left(P_i^2\right)'(-i)\right| = 4\sqrt{2} > 1,$$

and so this 2-cycle is repelling.

(b) The numbers $e^{2\pi i/7}$, $e^{4\pi i/7}$, $e^{8\pi i/7}$ form a 3-cycle of P_0, with multiplier

$$\begin{aligned} \left(P_0^3\right)'\left(e^{2\pi i/7}\right) &= P_0'\left(e^{2\pi i/7}\right)P_0'\left(e^{4\pi i/7}\right)P_0'\left(e^{8\pi i/7}\right) \\ &= \left(2e^{2\pi i/7}\right)\left(2e^{4\pi i/7}\right)\left(2e^{8\pi i/7}\right) \\ &= 8, \end{aligned}$$

by Theorem 2.4. Hence

$$\left|\left(P_0^3\right)'\left(e^{2\pi i/7}\right)\right| > 1,$$

and so this 3-cycle is repelling.

(c) The numbers $\frac{1}{2}(-1+\sqrt{2})$, $\frac{1}{2}(-1-\sqrt{2})$ form a 2-cycle of $P_{-5/4}$, with multiplier

$$\begin{aligned} \left(P^2_{-5/4}\right)'\left(\tfrac{1}{2}(-1+\sqrt{2})\right) &= P'_{-5/4}\left(\tfrac{1}{2}(-1+\sqrt{2})\right) \\ &\quad \times P'_{-5/4}\left(\tfrac{1}{2}(-1-\sqrt{2})\right) \\ &= (-1+\sqrt{2})(-1-\sqrt{2}) \\ &= -1, \end{aligned}$$

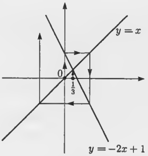
by Theorem 2.4. Hence

$$\left|\left(P^2_{-5/4}\right)'\left(\tfrac{1}{2}(-1+\sqrt{2})\right)\right| = 1,$$

and so this 2-cycle is indifferent.

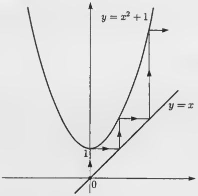
Section 3

3.1 **(a)**



(b) If $x_0 = 0$, then $\{x_n\}$ tends to infinity and if $x_0 = \frac{1}{3}$, then the sequence is constant. In Problem 1.9(b)(iii) we found that $z_{n+1} = az_n + b$, $n = 0, 1, 2, \ldots$, tends to infinity if $|a| > 1$ unless $z_0 = b/(1-a)$. Here we have $a = -2$, $b = 1$ and $b/(1-a) = \frac{1}{3}$, and so our answers agree with this result.

3.2 **(a)** With $x_0 = 0$, graphical iteration gives the following diagram.



(b) Since

$$\begin{aligned} x^2 + 1 > x &\iff x^2 - x + 1 > 0 \\ &\iff \left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4} > 0, \end{aligned}$$

we deduce that

$$x^2 + 1 > x, \quad \text{for all } x \in \mathbb{R}.$$

Hence the graph $y = x^2 + 1$ lies strictly above $y = x$, that is, the function $f(x) = x^2 + 1$ has no real fixed points. Also $x_{n+1} = x_n^2 + 1 > x_n$ and so the sequence $\{x_n\}$ is increasing and must tend to infinity (because there are no fixed points to prevent this).

(c) Since $x_n = P_1^n(x_0)$, for $n = 1, 2, \ldots$, we deduce that

$$P_1^n(x_0) \to \infty \text{ as } n \to \infty, \quad \text{for all } x_0 \in \mathbb{R},$$

and so no point of $\mathbb{R}$ belongs to K_1.

3.3 (a) If c is real and y is real, then

$$P_c(iy) = (iy)^2 + c = -y^2 + c,$$

which is real.

(b) Theorem 3.1 implies that if $c > \frac{1}{4}$, then

$$P_c^n(x) \to \infty \text{ as } n \to \infty, \quad \text{for all } x \in \mathbb{R},$$

and hence, by part (a), that

$$P_c^n(P_c(iy)) \to \infty \text{ as } n \to \infty, \quad \text{for all } y \in \mathbb{R}.$$

Thus

$$P_c^{n+1}(iy) \to \infty \text{ as } n \to \infty, \quad \text{for all } y \in \mathbb{R},$$

and so no point of the imaginary axis belongs to K_c.

3.4 If $c < -2$ and $y \in \mathbb{R}$, then

$$\begin{aligned} P_c(iy) &= -y^2 + c \\ &\leq c \\ &< -\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c} \end{aligned}$$

and so $P_c(iy)$ lies outside the interval I_c (see Figure 3.7). Hence

$$P_c^n(P_c(iy)) \to \infty \text{ as } n \to \infty, \quad \text{for all } y \in \mathbb{R},$$

by (3.3), and so

$$P_c^{n+1}(iy) \to \infty \text{ as } n \to \infty, \quad \text{for all } y \in \mathbb{R}.$$

Thus no point of the imaginary axis belongs to K_c.

Section 4

4.1 In Problem 3.4 we saw that, if $c < -2$, then K_c does not meet the imaginary axis. Since K_c has points in both $G_1 = \{z : \operatorname{Re} z > 0\}$ and $G_2 = \{z : \operatorname{Re} z < 0\}$ (for example, the fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$), we deduce that K_c is disconnected for $c < -2$.

4.2 (a) By Theorem 3.2, K_c contains 0 if $c \in \left[-2, \frac{1}{4}\right]$ and so, by Theorem 4.1, K_c is connected and hence $c \in M$.

(b) Since K_c is disconnected for $c > \frac{1}{4}$ and $c < -2$, by Problem 4.1 and the discussion preceding it, we have $c \notin M$ for $c > \frac{1}{4}$ and $c < -2$. Hence, by part (a), $M \cap \mathbb{R} = \left[-2, \frac{1}{4}\right]$.

4.3 (a) If $c = -2$, then the terms of $\{P_c^n(0)\}$ are

$$-2, 2, 2, \ldots .$$

Since all these terms lie in $\{z : |z| \leq 2\}$, we deduce by Corollary 1 that $-2 \in M$.

(b) If $c = 1 + i$, then the terms of $\{P_c^n(0)\}$ are

$$1 + i, \ (1 + i)^2 + 1 + i = 1 + 3i, \ \ldots .$$

Since $|1 + 3i| = \sqrt{10} > 2$, we deduce by Corollary 1 that $1 + i \notin M$.

(c) If $c = i$, then the terms of $\{P_c^n(0)\}$ are

$$i, -1 + i, -i, -1 + i, \ldots .$$

Since all these terms lie in $\{z : |z| \leq 2\}$, we deduce by Corollary 1 that $i \in M$.

(d) If $c = \sqrt{2}\, i$, then the terms of $\{P_c^n(0)\}$ are

$$\sqrt{2}\, i, -2 + \sqrt{2}\, i, \ldots .$$

Since $|-2 + \sqrt{2}\, i| = \sqrt{6} > 2$, we deduce by Corollary 1 that $\sqrt{2}\, i \notin M$.

4.4 (a) The point $c = -0.9 + 0.1i$ appears to lie in the disc $|z + 1| < \frac{1}{4}$, so we use Theorem 4.4(b). Since

$$|c + 1| = |0.1 + 0.1i| = 0.1414\ldots < \tfrac{1}{4},$$

P_c has an attracting 2-cycle, by Theorem 4.4(b). Thus c lies in M, by Theorem 4.3.

(b) The point $c = 0.2 + 0.5i$ appears to lie inside the cardioid, so we use Theorem 4.4(a). Since $|c|^2 = 0.29$ and $\operatorname{Re} c = 0.2$, we have

$$\begin{aligned} \left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8 \operatorname{Re} c &= (2.32 - 1.5)^2 + 1.6 \\ &= 2.2724 < 3, \end{aligned}$$

and so P_c has an attracting fixed point, by Theorem 4.4(a). Thus c lies in M, by Theorem 4.3.

4.5 (a) Since $P_c^2(z) = \left(z^2 + c\right)^2 + c$, we have

$$P_c^2(z) - z = z^4 + 2cz^2 - z + c^2 + c.$$

Also

$$\begin{aligned} (P_c(z) - z)\left(z^2 + z + c + 1\right) &= \left(z^2 - z + c\right)\left(z^2 + z + c + 1\right) \\ &= z^4 + 2cz^2 - z + c^2 + c, \end{aligned}$$

as required.

(b) The 2-cycles of P_c are the solutions of $P_c^2(z) - z = 0$ which are not solutions of $P_c(z) - z = 0$. Hence, by part (a), they are the solutions of

$$z^2 + z + c + 1 = 0;$$

which gives the 2-cycle α_1, α_2 where

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{-\tfrac{3}{4} - c}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{-\tfrac{3}{4} - c}.$$

Note that $c = -\frac{3}{4}$ must be excluded, because in that case $\alpha_1 = \alpha_2 = -\frac{1}{2}$, which is a fixed point of $P_{-3/4}$. By Theorem 2.4, the multiplier of this 2-cycle is

$$\begin{aligned} \left(P_c^2\right)'(\alpha_1) &= P_c'(\alpha_1) P_c'(\alpha_2) \\ &= (2\alpha_1)(2\alpha_2) \\ &= 4\alpha_1\alpha_2. \end{aligned}$$

(c) Since $\alpha_1\alpha_2 = c + 1$, we deduce that the above 2-cycle is attracting if and only if

$$\left|\left(P_c^2\right)'(\alpha_1)\right| = |4\alpha_1\alpha_2| = 4|c + 1| < 1;$$

that is, if and only if $|c + 1| < \frac{1}{4}$, as required.

4.6 The function P_c has a super-attracting 3-cycle if and only if

$$P_c^3(0) = \left(c^2 + c\right)^2 + c = 0,$$

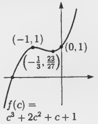
but $P_c(0) = c \neq 0, P_c^2(0) = c^2 + c \neq 0$. Thus we seek the solutions of

$$c^4 + 2c^3 + c^2 + c = 0$$

which are not $c = 0$ nor $c = -1$. Since $c = 0$ is a solution and $c = -1$ is not, we need to solve

$$c^3 + 2c^2 + c + 1 = 0.$$

Using calculus, we see that the real function $f(c) = c^3 + 2c^2 + c + 1$ has the following graph, so that there is only one real zero.



Since $f(-1.8) = -0.152$ and $f(-1.7) = 0.167$, it follows that this value of c lies in $[-1.8, -1.7]$.

The remaining two solutions form a pair of complex conjugates. (In fact the three solutions, correct to three decimal places, are -1.755 and $-0.123 \pm 0.745i$. (See Figure 2.18(b)!) These solutions may be obtained by using Formula (0.6) in the Introduction to *Unit A1*!)

4.7 (a) Since any solution of $P_c(z) - z = 0$ is also a solution of $P_c^3(z) - z = 0$, we expect $P_c(z) - z$ to be a factor of $P_c^3(z) - z = 0$. By inspection, we find that

$$\begin{aligned} P_c^3(z) - z &= (P_c(z) - z)Q_c(z) \\ &= \left(z^2 - z + c\right) Q_c(z), \end{aligned}$$

where

$$\begin{aligned} Q_c(z) = z^6 + z^5 &+ (3c+1)z^4 + (2c+1)z^3 \\ &+ \left(3c^2 + 3c + 1\right) z^2 + \left(c^2 + 2c + 1\right) z \\ &+ c^3 + 2c^2 + c + 1. \end{aligned}$$

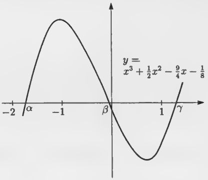
(Begin by finding the coefficient of z^6 and the constant term, and then work inwards from both ends, obtaining the coefficient of z^3 twice, as a check.)

(b) On substituting $c = -\frac{7}{4}$ we obtain, after some arithmetic,

$$\begin{aligned} Q_{-7/4}(z) &= z^6 + z^5 - \tfrac{17}{4}z^4 - \tfrac{5}{2}z^3 + \tfrac{79}{16}z^2 + \tfrac{9}{16}z + \tfrac{1}{64} \\ &= \left(z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8}\right)^2, \end{aligned}$$

by inspection again.

(c) The graph $y = x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8}$ is as follows.



Hence

$$z^3 + \tfrac{1}{2}z^2 - \tfrac{9}{4}z - \tfrac{1}{8} = (z - \alpha)(z - \beta)(z - \gamma), \qquad (1)$$

where α, β, γ are the real zeros in the figure. Since these are the only solutions of $P_{-7/4}^3(z) - z = 0$, which are not fixed points of $P_{-7/4}$, they must form a 3-cycle of $P_{-7/4}$. By Theorem 2.4, the multiplier of this 3-cycle is

$$\begin{aligned} \left(P_c^3\right)'(\alpha) &= P_c'(\alpha)P_c'(\beta)P_c'(\gamma) \\ &= (2\alpha)(2\beta)(2\gamma) \\ &= 8\alpha\beta\gamma \\ &= 1, \end{aligned}$$

since $\alpha\beta\gamma = \frac{1}{8}$, by Equation (1).

Hence, by Theorem 4.6(a), a saddle-node bifurcation occurs at $c = -7/4$, so we expect to see a small cardioid-shaped periodic region with cusp at this point. This is included in Figure 4.8, and its centre is at the point $c = -1.755$, found in Problem 4.6.

4.8 Since

$$P_c(\zeta) = \zeta^2 + c = \zeta \quad \left(c = \zeta - \zeta^2\right),$$

we find that ζ is a fixed point of P_c. Also, the multiplier is

$$P_c'(\zeta) = 2\zeta,$$

which is a root of unity ($\neq 1$). Hence, by Theorem 4.6(b), a period-multiplying bifurcation occurs at c. (In fact, since the cardioid is the image of the circle $|z| = \frac{1}{2}$ under the function $f(z) = z - z^2$, the point c lies on the cardioid.)

If $\zeta = -\frac{1}{2}$, then $2\zeta = -1$, which is a primitive square root of unity and so a period-doubling bifurcation occurs at

$$c = -\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)^2 = -\tfrac{3}{4},$$

and this is visible in Figure 4.8.

If $\zeta = \frac{1}{2}e^{2\pi i/3}$, then $2\zeta = e^{2\pi i/3}$, which is a primitive cube root of unity and so a 'period-trebling' bifurcation occurs at

$$c = \tfrac{1}{2}e^{2\pi i/3} - \left(\tfrac{1}{2}e^{2\pi i/3}\right)^2 = -\tfrac{1}{8} + \tfrac{3}{8}\sqrt{3}i,$$

and this is visible in Figure 4.8.

If $\zeta = \frac{1}{2}i$, then $2\zeta = i$, which is a primitive fourth root of unity and so a 'period-quadrupling' bifurcation occurs at

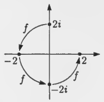
$$c = \tfrac{1}{2}i - \left(\tfrac{1}{2}i\right)^2 = \tfrac{1}{4} + \tfrac{1}{2}i,$$

and this is visible in Figure 4.8.

SOLUTIONS TO THE EXERCISES

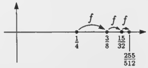
Section 1

1.1 (a) $z_0 = 2i, z_1 = -2, z_2 = -2i, z_3 = 2.$



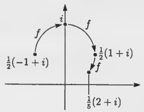
Here $f(z) = iz$.

(b) $z_0 = \frac{1}{4}, z_1 = \frac{3}{8}, z_2 = \frac{15}{32}, z_4 = \frac{255}{512}.$



Here $f(z) = 2z(1-z)$.

(c) $z_0 = \frac{1}{2}(-1+i), z_1 = i, z_2 = \frac{1}{2}(1+i), z_3 = \frac{1}{5}(2+i).$



Here $f(z) = \dfrac{z}{z+1}$.

1.2 (a) We have to solve $f(z) = z$:

$$z - z^2 = z \iff z^2 = 0 \iff z = 0.$$

Since $f'(z) = 1 - 2z$, we have

$$f'(0) = 1,$$

so the fixed point at 0 is indifferent.

(b) We have to solve $f(z) = z$:

$$2z(1-z) = z \iff z(1-2z) = 0,$$

so the fixed points of f are at 0 and $\frac{1}{2}$. Since $f'(z) = 2 - 4z$, we have

$$f'(0) = 2 \quad \text{and} \quad f'\left(\tfrac{1}{2}\right) = 0.$$

Thus the fixed point at 0 is repelling, whereas that at $\frac{1}{2}$ is super-attracting.

(c) We have to solve $f(z) = z$:

$$z^2 - \tfrac{1}{2} = z \iff z^2 - z - \tfrac{1}{2} = 0,$$

so the fixed points of f are at $\frac{1}{2}(1 \pm \sqrt{3})$. Since $f'(z) = 2z$, we have

$$\left|f'\left(\tfrac{1}{2}(1+\sqrt{3})\right)\right| = 1 + \sqrt{3} = 2.732\ldots$$

and

$$\left|f'\left(\tfrac{1}{2}(1-\sqrt{3})\right)\right| = 0.732\ldots.$$

Thus the fixed point of f at $\frac{1}{2}(1+\sqrt{3})$ is repelling, whereas that at $\frac{1}{2}(1-\sqrt{3})$ is attracting.

(d) We have to solve $f(z) = z$:

$$\frac{z}{z+1} = z \iff z = 0,$$

so the only fixed point of f is at 0. Since $f'(z) = 1/(z+1)^2$, we have

$$f'(0) = 1.$$

Thus the fixed point of f at 0 is indifferent.

1.3 Putting $w_n = h(z_n) = 1 - 2z_n$ for $n = 0, 1, 2, \ldots$, we have

$$z_n = \tfrac{1}{2}(1 - w_n), \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence the given sequence is conjugate to the sequence $\{w_n\}$, where

$$\begin{aligned}\tfrac{1}{2}(1 - w_{n+1}) &= 2\left(\tfrac{1}{2}(1-w_n)\right)\left(1 - \tfrac{1}{2}(1-w_n)\right)\\ &= \tfrac{1}{2}(1-w_n)(1+w_n)\\ &= \tfrac{1}{2}(1-w_n^2),\end{aligned}$$

so that $w_{n+1} = w_n^2$, for $n = 0, 1, 2, \ldots$, as required. Since $w_n = w_0^{2^n}$, for $n = 0, 1, 2, \ldots$, by Example 1.2(b), we deduce that

$$\begin{aligned}z_n &= \tfrac{1}{2}(1 - w_n)\\ &= \tfrac{1}{2}\left(1 - w_0^{2^n}\right)\\ &= \tfrac{1}{2}\left(1 - (1-2z_0)^{2^n}\right), \quad \text{for } n = 0, 1, 2, \ldots.\end{aligned}$$

1.4 To find the fixed points of f we solve $f(z) = z$:

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0,$$

so the fixed points of f are at

$$\frac{1}{2c}\left(a - d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

(a) If $(a-d)^2 + 4bc = 0$, then f has just one fixed point at $\alpha = (a-d)/(2c)$. Since ∞ is not a fixed point of $\widehat{f}$ (because $c \neq 0$), the only fixed point of $\widehat{f}$ is at α. Putting $w_n = h(z_n) = 1/(z_n - \alpha)$, we have $z_n = \alpha + 1/w_n$, for $n = 0, 1, 2, \ldots$.

Hence the sequence $\{z_n\}$ is conjugate to the sequence $\{w_n\}$, where

$$\begin{aligned}\alpha + 1/w_{n+1} &= \frac{a(\alpha + 1/w_n) + b}{c(\alpha + 1/w_n) + d}\\ &= \frac{a\alpha + b + a/w_n}{c\alpha + d + c/w_n}\\ &= \frac{(a\alpha + b)w_n + a}{(c\alpha + d)w_n + c}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{w_{n+1}} &= \frac{(a\alpha + b)w_n + a - \alpha(c\alpha + d)w_n - c\alpha}{(c\alpha + d)w_n + c}\\ &= \frac{a - c\alpha}{(c\alpha + d)w_n + c} \quad \left(\text{since } \frac{a\alpha + b}{c\alpha + d} = \alpha\right),\end{aligned}$$

and so

$$\begin{aligned}w_{n+1} &= \left(\frac{c\alpha + d}{a - c\alpha}\right)w_n + \frac{c}{a - c\alpha}\\ &= w_n + 2c/(a+d), \quad \text{for } n = 0, 1, 2, \ldots, \qquad (1)\end{aligned}$$

because $c\alpha + d = \frac{1}{2}(a+d) = a - c\alpha$. Note that $c \neq 0$ and also $a + d \neq 0$, because

$$\begin{aligned}(a+d)^2 &= (a-d)^2 + 4bc + 4(ad - bc)\\ &= 4(ad - bc) \neq 0.\end{aligned}$$

It follows from (1) that $w_n \to \infty$ as $n \to \infty$, for all $w_0 \in \widehat{\mathbb{C}}$ and hence

$$z_n = \alpha + \frac{1}{w_n} \to \alpha \text{ as } n \to \infty, \quad \text{for all } z_0 \in \widehat{\mathbb{C}}.$$

Remark The sequence w_n tends to infinity along a straight line, and so z_n tends to α along a circle; see the solution to Exercise 1.1(c).

(b) If $(a-d)^2 + 4bc \neq 0$, then f has the two fixed points

$$\alpha = \frac{1}{2c}\left(a - d + \sqrt{(a-d)^2 + 4bc}\right)$$

and

$$\beta = \frac{1}{2c}\left(a - d - \sqrt{(a-d)^2 + 4bc}\right).$$

Putting $w_n = h(z_n) = (z_n - \alpha)/(z_n - \beta)$, we have $z_n = (-\beta w_n + \alpha)/(-w_n + 1)$, for $n = 0, 1, 2, \ldots$. Hence the sequence $\{z_n\}$ is conjugate to the sequence

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(f(z_n)) \\ &= \left(h \circ f \circ h^{-1}\right)(w_n). \end{aligned}$$

Composing the Möbius transformations h, f and h^{-1} (for example, by multiplying the corresponding 2×2 matrices), we obtain

$$w_{n+1} = \frac{(-a\beta - b + c\alpha\beta + d\alpha)w_n + \left(a\alpha + b - c\alpha^2 - d\alpha\right)}{\left(-a\beta - b + c\beta^2 + d\beta\right)w_n + (a\alpha + b - c\alpha\beta - d\beta)}.$$

Since $a\alpha + b = \alpha(c\alpha + d)$ and $a\beta + b = \beta(c\beta + d)$, it follows that

$$\begin{aligned} w_{n+1} &= \frac{(\alpha - \beta)(c\beta + d)w_n + 0}{0 + (\alpha - \beta)(c\alpha + d)} \\ &= \left(\frac{c\beta + d}{c\alpha + d}\right) w_n, \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned}$$

Now

$$f'(\alpha) = \frac{ad - bc}{(c\alpha + d)^2},$$

and so to obtain $w_{n+1} = f'(\alpha)w_n$, for $n = 0, 1, 2, \ldots$, we need to check that $ad - bc = (c\alpha + d)(c\beta + d)$. Since

$$c\alpha + d = \tfrac{1}{2}\left(a + d + \sqrt{(a-d)^2 + 4bc}\right)$$

and

$$c\beta + d = \tfrac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right),$$

we obtain

$$\begin{aligned} (c\alpha + d)(c\beta + d) &= \tfrac{1}{4}\left((a+d)^2 - (a-d)^2 - 4bc\right) \\ &= ad - bc, \end{aligned}$$

as required.

If $|f'(\alpha)| < 1$, then $w_n \to 0$ as $n \to \infty$, for all $w_0 \in \mathbb{C}$, and hence

$$z_n = \frac{-\beta w_n + \alpha}{-w_n + 1} \to \alpha \text{ as } n \to \infty,$$

for all $z_0 \in \widehat{\mathbb{C}} - \{\beta\}$.

If $|f'(\alpha)| > 1$, then $w_n \to \infty$ as $n \to \infty$, for all $w_0 \in \mathbb{C} - \{0\}$ and hence

$$z_n = \frac{-\beta w_n + \alpha}{-w_n + 1} \to \beta \text{ as } n \to \infty,$$

for all $z_0 \in \widehat{\mathbb{C}} - \{\alpha\}$.

If $|f'(\alpha)| = 1$, then $|w_n| = |w_0|$, for $n = 1, 2, \ldots$, and so the sequence $\{w_n\}$ remains on the circle with centre 0 and radius $|w_0|$. Hence the sequence $\{z_n\}$ remains on the image of this circle under h^{-1}, which is a generalized circle with α and β as inverse points.

Section 2

2.1 Using Theorem 2.1, we find that

$$z_{n+1} = 3z_n(1 - z_n) = -3z_n^2 + 3z_n, \quad n = 0, 1, 2, \ldots,$$

is conjugate to

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $w_n = -3z_n + 3/2$, for $n = 0, 1, 2, \ldots$, and

$$d = -3 \times 0 + 3/2 - 9/4 = -3/4.$$

2.2 **(a)** If $|c| \le \frac{1}{4}$ and $|z| \le \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, then

$$\begin{aligned} |P_c(z)| &= \left|z^2 + c\right| \\ &\le |z|^2 + |c| \quad \text{(Triangle Inequality)} \\ &\le \left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right)^2 + |c| \\ &= \tfrac{1}{4} + \sqrt{\tfrac{1}{4} - |c|} + \tfrac{1}{4} - |c| + |c| \\ &= \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}, \end{aligned}$$

as required.

(b) If $|c| \le \frac{1}{4}$ and $|z| \le \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$, then, by part (a) applied repeatedly,

$$|P_c^n(z)| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}, \quad \text{for } n = 0, 1, 2, \ldots,$$

so that $P_c^n(z) \not\to \infty$ as $n \to \infty$, and hence $z \notin E_c$.

Therefore, if $|c| \le \frac{1}{4}$, then we have

$$\left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right\} \subseteq K_c,$$

as required.

(c) If $|c| \le \frac{1}{4}$, then, by part (b) and Theorem 2.3(a),

$$\left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - |c|}\right\} \subseteq K_c \subseteq \left\{z : |z| \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + |c|}\right\}.$$

Now, if c is close to 0, then $\frac{1}{2} + \sqrt{\frac{1}{4} \pm |c|}$ are both close to 1, and so K_c is approximately equal to the closed unit disc.

2.3 **(a)** Since

$$f(i) = -i \quad \text{and} \quad f(-i) = i,$$

the point $\alpha = i$ is periodic with period 2, and belongs to the 2-cycle i, $-i$. Since $f'(z) = -1$, the multiplier of this 2-cycle is, by Theorem 2.4,

$$f'(i)f'(-i) = (-1) \times (-1) = 1.$$

Thus this 2-cycle is indifferent, and so i is an indifferent periodic point of f with period 2.

(b) Since

$$f\left(\tfrac{1}{2}(-1 + \sqrt{5})\right) = \tfrac{1}{2}(-1 - \sqrt{5})$$

and

$$f\left(\tfrac{1}{2}(-1 - \sqrt{5})\right) = \tfrac{1}{2}(-1 + \sqrt{5}),$$

the point $\alpha = \frac{1}{2}(-1 + \sqrt{5})$ is periodic with period 2, and belongs to the 2-cycle $\frac{1}{2}(-1 + \sqrt{5})$, $\frac{1}{2}(-1 - \sqrt{5})$. Since $f'(z) = 2z$, the multiplier of this 2-cycle is, by Theorem 2.4,

$$\begin{aligned} f'\left(\tfrac{1}{2}(-1 + \sqrt{5})\right) f'\left(\tfrac{1}{2}(-1 - \sqrt{5})\right) &= (-1 + \sqrt{5})(-1 - \sqrt{5}) \\ &= -4. \end{aligned}$$

Thus this 2-cycle is repelling, and so $\frac{1}{2}(-1 + \sqrt{5})$ is a repelling periodic point of f with period 2.

(c) Since

$$f(0) = i \quad \text{and} \quad f(i) = 0,$$

the point $\alpha = 0$ is periodic with period 2, and belongs to the 2-cycle 0, i. Since $f'(z) = 3z^2$, the multiplier of this 2-cycle is, by Theorem 2.4,

$$f'(0)f'(i) = 0.$$

Thus this 2-cycle is super-attracting, and so 0 is a super-attracting periodic point of f with period 2.

(d) Since

$$f\left(e^{\pi i/13}\right) = e^{3\pi i/13}, \quad f\left(e^{3\pi i/13}\right) = e^{9\pi i/13},$$

and

$$f\left(e^{9\pi i/13}\right) = e^{27\pi i/13} = e^{\pi i/13},$$

the point $\alpha = e^{\pi i/13}$ is periodic with period 3, and belongs to the 3-cycle $e^{\pi i/13}$, $e^{3\pi i/13}$, $e^{9\pi i/13}$. Since $f'(z) = 3z^2$, the multiplier of the 3-cycle is, by Theorem 2.4,

$$\begin{aligned} &f'\left(e^{\pi i/13}\right) f'\left(e^{3\pi i/13}\right) f'\left(e^{9\pi i/13}\right) \\ &= \left(3e^{2\pi i/13}\right)\left(3e^{6\pi i/13}\right)\left(3e^{18\pi i/13}\right) \\ &= 27e^{26\pi i/13} \\ &= 27. \end{aligned}$$

Thus the 3-cycle is repelling, and so $e^{\pi i/13}$ is a repelling periodic point of f with period 3.

2.4 (a) Since $P_0^n(z) = z^{2^n}$, α is a periodic point of P_0 if and only if it satisfies the equation $z^{2^p} = z$, for some positive integer p. Now

$$z^{2^p} = z \iff z\left(z^{2^p-1} - 1\right) = 0.$$

Thus either $\alpha = 0$, which is a super-attracting fixed point of P_0, or α is a $(2^p - 1)$th root of unity. Since $P_0'(z) = 2z$, the multiplier of α has absolute value 2^q for some factor q of p and α is therefore repelling. Thus the repelling periodic points of P_0 are the $(2^p - 1)$th roots of unity, for $p = 1, 2, \ldots$.

Now, if Γ is an arc of the unit circle which subtends an angle $\theta > 0$ at 0, then we can choose p so large that

$$\frac{2\pi}{2^p - 1} < \theta,$$

and this guarantees that at least one of the $(2^p - 1)$th roots of unity (which are evenly spaced around the unit circle) lies in Γ.

(b) Since $P_0^n(z) = z^{2^n}$, α is a backward iterate of 1 under P_0 if and only if it satisfies the equation

$$z^{2^n} = 1,$$

for some positive integer n, that is, if and only if α is a 2^nth root of unity for some positive integer n.

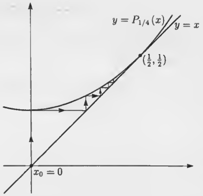
Now, if Γ is an arc of the unit circle which subtends an angle $\theta > 0$ at 0, then we can choose n so large that

$$\frac{2\pi}{2^n} < \theta,$$

and this guarantees that at least one of the 2^nth roots of unity lies in Γ.

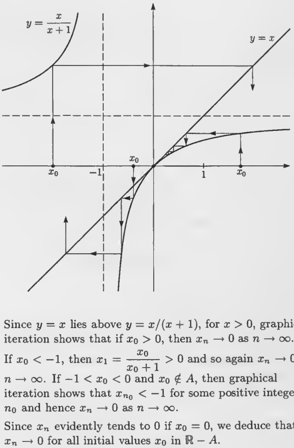
Section 3

3.1



The iteration sequence $\{x_n\}$ converges to the indifferent fixed point $\frac{1}{2}$ of $P_{1/4}$.

3.2 (a) First we plot $y = x$ and $y = \dfrac{x}{x+1}$ on the same diagram, and apply graphical iteration with various initial points x_0.



Since $y = x$ lies above $y = x/(x + 1)$, for $x > 0$, graphical iteration shows that if $x_0 > 0$, then $x_n \to 0$ as $n \to \infty$. If $x_0 < -1$, then $x_1 = \dfrac{x_0}{x_0 + 1} > 0$ and so again $x_n \to 0$ as $n \to \infty$. If $-1 < x_0 < 0$ and $x_0 \notin A$, then graphical iteration shows that $x_{n_0} < -1$ for some positive integer n_0 and hence $x_n \to 0$ as $n \to \infty$.

Since x_n evidently tends to 0 if $x_0 = 0$, we deduce that $x_n \to 0$ for all initial values x_0 in $\mathbb{R} - A$.

(b) The function

$$f(z) = \frac{z}{z+1}$$

is a Möbius transformation, with $a = 1$, $b = 0$, $c = 1$, $d = 1$, which satisfies $(a - d)^2 + 4bc = 0$. Hence, by Exercise 1.4(a), $\alpha = 0$ is the only fixed point of f and the iteration sequence

$$z_{n+1} = \hat{f}(z_n), \quad n = 0, 1, 2, \ldots,$$

converges to $\alpha = 0$ for all initial points z_0 in $\hat{\mathbb{C}}$. The result in part (a) is therefore a special case of Exercise 1.4(a).

Section 4

4.1 (a) Since $|P_c(0)| = |c| = \sqrt{5} > 2$ if $c = -1 + 2i$, we deduce, by Corollary 1 to Theorem 4.1, that $c \notin M$.

(b) Since $P_{-1}(0) = -1$ and $P_{-1}(-1) = 0$, we deduce that the terms of the sequence $\{P_{-1}^n(0)\}$ are $0, -1, 0, -1, \ldots$, and so

$$|P_{-1}^n(0)| \leq 2, \text{ for } n = 1, 2, \ldots.$$

Hence $-1 \in M$.

(c) For $c = -1 + i$,

$$P_c(0) = -1 + i,$$
$$P_c^2(0) = (-1+i)^2 + (-1+i) = -1 - i,$$
$$P_c^3(0) = (-1-i)^2 + (-1+i) = -1 + 3i.$$

Since

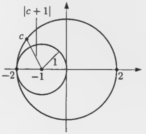
$$\left|P_c^3(0)\right| = \sqrt{10} > 2,$$

we deduce, by Corollary 1 to Theorem 4.1, that $c \notin M$.

4.2 Suppose that $|c| = 2$ but $c \neq -2$. Then

$$\left|c^2 + c\right| = |c(c+1)| = |c|\,|c+1| = 2|c+1|.$$

Now $|c+1|$ is the distance from c to -1, and so $|c+1| > 1$.



Hence $\left|c^2 + c\right| = \left|P_c^2(0)\right| > 2$ and so, by Corollary 1 to Theorem 4.1, $c \notin M$.

4.3 (a) Since $|c| > r_c$, if $|c| > 2$, we deduce by Theorem 2.2 that if $|c| > 2$, then the sequence

$$|P_c^n(c)| = \left|P_c^{n+1}(0)\right|, \quad n = 0, 1, 2, \ldots,$$

is increasing and so all its terms exceed 2. Hence

$$M_n \subseteq \{c : |c| \leq 2\} = M_1, \quad \text{for } n = 2, 3, \ldots.$$

(b) Since $r_c \leq 2$, if $|c| \leq 2$, we deduce by Theorem 2.2 that if $|c| \leq 2$ and $|P_c^n(0)| > 2$, then

$$\left|P_c^{n+1}(0)\right| = |P_c(P_c^n(0))| \geq |P_c^n(0)| > 2.$$

Thus, for $|c| \leq 2$, we deduce that

$$\left|P_c^{n+1}(0)\right| \leq 2 \implies |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \ldots,$$

that is, $M_{n+1} \subseteq M_n$, for $n = 1, 2, \ldots$, as required.

4.4 (a) If $c = -1.1 - 0.1i$, then

$$|c+1| = |-0.1 - 0.1i| = 0.1414\ldots < \tfrac{1}{4}.$$

Hence, by Theorem 4.4(b), P_c has an attracting 2-cycle and so, by Theorem 4.3, $c \in M$.

(b) If $c = 0.6i$, then $|c|^2 = 0.36$ and $\operatorname{Re} c = 0$, so that

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\operatorname{Re} c = (8 \times 0.36 - 1.5)^2$$
$$= 1.9044 < 3.$$

Hence, by Theorem 4.4(a), P_c has an attracting fixed point and so, by Theorem 4.3, $c \in M$.

4.5 (a) If some point α, say, of $\partial\mathcal{R}$ lies outside M, then because M is closed there is an open disc D with centre α which lies entirely outside M. Since $\mathcal{R} \subseteq M$, the open disc D does not meet $\mathcal{R}$, and this contradicts the fact that α is a boundary point of $\mathcal{R}$. Hence $\partial\mathcal{R} \subseteq M$.

(b) (i) From Figure 4.12, it appears that the point $c = \frac{1}{4} + \frac{1}{2}i$ lies on the cardioid with parametrization

$$\gamma(t) = \tfrac{1}{2}e^{it} - \tfrac{1}{4}e^{2it} \quad (t \in [-\pi, \pi]).$$

To verify this, either note that

$$\gamma(\pi/2) = \tfrac{1}{2}i - \tfrac{1}{4}(-1) = \tfrac{1}{4} + \tfrac{1}{2}i,$$

or check that c satisfies

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\operatorname{Re} c = 3.$$

Thus c lies on the boundary of the periodic region in M given by Theorem 4.4(a), and hence $c \in M$ by part (a).

(ii) The point $c = -1 + \frac{1}{4}i$ lies on the circle $|c+1| = \frac{1}{4}$. Thus c lies on the boundary of the periodic region in M given by Theorem 4.4(b), and hence $c \in M$ by part (a).

4.6 By Theorem 4.5, the function P_c has a super-attracting 4-cycle if and only if

$$P_c^4(0) = 0, \text{ but } P_c(0), P_c^2(0) \text{ and } P_c^3(0) \text{ are non-zero.}$$

Now $P_c(0) = c$, $P_c^2(0) = c^2 + c$, $P_c^3(0) = c^4 + 2c^3 + c^2 + c$ and $P_c^4(0)$ is of the form

$$P_c^4(0) = c^8 + 4c^7 + \cdots + c^2 + c.$$

If $P_c(0) = 0$ or $P_c^2(0) = 0$, then $P_c^4(0) = P_c^2(P_c^2(0)) = 0$, and so $P_c^4(0)$ can be factorized as follows:

$$P_c^4(0) = (c^2 + c)(c^6 + 3c^5 + \cdots + 1).$$

Thus the values of c for which P_c has a super-attracting 4-cycle are the solutions of the equation

$$c^6 + 3c^5 + \cdots + 1 = 0,$$

and there are at most six such (distinct) solutions. Hence, there are at most six values of c for which P_c has a super-attracting 4-cycle.

Remark In fact it can be shown that each of the polynomial functions $c \longmapsto P_c^n(0)$ has only simple zeros. This makes it possible to count the number of values of c for which P_c has a super-attracting p-cycle, for each positive integer p. In particular, there are exactly six values of c for which P_c has a super-attracting 4-cycle, as indicated in Figure 4.8.

4.7 For $c = -\frac{5}{4}$, P_c has the 2-cycle

$$\alpha_1 = -\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}, \quad \alpha_2 = -\tfrac{1}{2} - \sqrt{\tfrac{1}{2}},$$

(by Problem 2.7(c)) with multiplier

$$\left(P_{-5/4}^2\right)'(\alpha_1) = 4\alpha_1\alpha_2 = 4\left(-\tfrac{1}{2} + \sqrt{\tfrac{1}{2}}\right)\left(-\tfrac{1}{2} - \sqrt{\tfrac{1}{2}}\right)$$
$$= -1.$$

Since -1 is a primitive square root of unity, we deduce, by Theorem 4.6(b), that a period-doubling bifurcation occurs at $c = -\frac{5}{4}$.

In Figure 4.8, we see that the point $-\frac{5}{4}$ lies where a periodic region with period 2 and a periodic region with period 4 touch, so a period-doubling bifurcation is visible.